An Investigation of Quasilocal Systems in General Relativity

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Abstract

We propose a method to define and investigate finite size systems in general relativity in terms of their matter plus gravitational energy content. We achieve this by adopting a generic formulation, that involves the embedding of an arbitrary dimensional timelike worldsheet into an arbitrary dimensional spacetime, to a $2+2$ picture. In our case, the closed 2-dimensional spacelike surface $\mathbb{S}$, that is orthogonal to the 2-dimensional timelike worldsheet $\mathbb{T}$ at every point, encloses the system in question. The corresponding Raychaudhuri equation of $\mathbb{T}$ is interpreted as a thermodynamic relation for spherically symmetric systems in quasilocal thermodynamic equilibrium and leads to a work-energy relation for more generic systems that are in nonequilibrium.

In the case of equilibrium, our quasilocal thermodynamic potentials are directly related to standard quasilocal energy definitions given in the literature. Quasilocal thermodynamic equilibrium is obtained by minimizing the Helmholtz free energy written via the mean extrinsic curvature of $\mathbb{S}$. Moreover, without any direct reference to surface gravity, we find that the system comes into quasilocal thermodynamic equilibrium when $\mathbb{S}$ is located at a generalized apparent horizon. We present a first law and the corresponding worldsheet–constant temperature. Examples of the Schwarzschild, Friedmann–Lemaître and Lemaître–Tolman geometries are investigated and compared. Conditions for the quasilocal thermodynamic and hydrodynamic equilibrium states to coincide are also discussed, and a quasilocal virial relation is suggested as a potential application of this approach.

For the case of nonequilibrium, we first apply a transformation of the formalism of our previous notation so that one may keep track of the quasilocal observables and the null cone observables in tandem. We identify three null tetrad gauge conditions that result from the integrability conditions of $\mathbb{T}$ and $\mathbb{S}$. This guarantees that our quasilocal system is well defined. In the Raychaudhuri equation of $\mathbb{T}$, we identify the quasilocal charge densities corresponding to the rotational and nonrotational degrees of freedom, in addition to a relative work density related to tidal fields. We define the corresponding quasilocal charges that appear in our work-energy relation and which can potentially be exchanged with the surroundings. These charges and our tetrad conditions are invariant under the boosting of the observers in the direction orthogonal to $\mathbb{S}$. We apply our construction to a radiating Vaidya spacetime, a C-metric and the interior of a Lanczos-van Stockum dust metric. Delicate issues related to axially symmetric stationary spacetimes and possible extensions to the Kerr geometry are also discussed.

Acknowledgements

I would like to thank to my supervisor, David L. Wiltshire, for allowing me to work on the subjects that I have been really curious about. I believe not every PhD student is offered such freedom. His critical suggestions and careful reading of this manuscript are highly appreciated. I must also add that it is a pleasure to have met his genuine personality.

Dedicated to Antonio Salieri.

Contents

1 Introduction

In physics, what we ultimately want to understand is how things work. Those *things* in question, of course, changed over the history of science as we gathered larger amounts of physical data and invented/discovered better mathematics. From point objects to electromagnetic waves, from subatomic particles to their corresponding fields, the very definition of 'things' has changed over time. Also it was with the advent of thermodynamics and statistical mechanics that we started asking deeper questions by investigating ensembles of particles in a conscious way. The 'thing' under investigation is then referred to as a *system*. This is the first keyword of this thesis.

Moreover, in terms of the dynamics of the things, i.e., how/why things work, Newton's construction of classical mechanics was the basis of all subsequent mathematical physics. Later with the Lagrangian and Hamiltonian formulations we obtained a more physically and mathematically concrete understanding of the dynamics of things. From that point on the *energy* concept became very important in physics. In fact, it is interesting that the word energy was introduced in the literature by Young just a few decades before the advent of Hamilton's formulation [1]. Energy is the second keyword of the thesis.

Today, hundred years after Einstein presented his theory of gravity, general relativity (GR) is still lacking unique definitions of a *system* and *energy*. It is true that we have a good understanding of the dynamics of matter particles on a curved background. We can calculate their momentum and energy. However, general relativity is not a theory which solely investigates the objects on a pre-defined geometry. In GR, the spacetime *geometry* itself is the fundamental object which possesses energy. Geometry is our last keyword.

The search for conserved quantities in general relativity started from its earliest days and it has been a very hot topic ever since then. We know that the stress-energy tensor, $T_{\mu\nu}$, is *locally* conserved, i.e., $D_\mu T^\mu{}_\nu = 0$ due to the Bianchi identities. However, this tensor is defined *only for the matter fields*. Then one can ask whether there is an analogous energy-momentum tensor that incorporates the effect of gravitational fields. In the late 1950s Bel [2] came up with a concrete proposal by defining a 4^{th}-rank tensor, $t_{\mu\nu\alpha\beta}$. This traceless tensor, $t_{\mu\nu\alpha\beta}$, is defined via the Weyl tensor and is analogous to the electromagnetic stress-energy tensor that is constructed from the electromagnetic field. It can be shown that in general relativity the Bel tensor is divergence-free. However, this holds *only in vacuum*.

For many researchers, the actual aim has been to find a *total* stress-energy tensor which includes the effect of both matter and gravitational fields, that is locally conserved. Accordingly, much research interest has focused on pseudotensors [3, 4, 5, 6, 7]. In particular, suppose we define the total stress-energy tensor via $\mathcal{T}_{\mu\nu} = T_{\mu\nu} + \tau_{\mu\nu}$, where $\tau_{\mu\nu}$ is the object that carries information about the gravitational energy-momentum content. For various definitions of $\tau_{\mu\nu}$ it can be shown that $D_\mu \mathcal{T}^\mu_\nu = 0$[1]. However, this can be achieved for those $\tau_{\mu\nu}$ which *are not tensors*, but rather *pseudotensors*. This means that the resulting conservation law is coordinate dependent which is an unwanted property for a covariant theory.

In classical field theories, our understanding of conserved quantities is directly related to the corresponding continues symmetries on account of Noether's Theorem. For example, in simple words, if a system has rotational symmetry then the angular momentum is conserved and if it has time symmetry then the total energy of the system is conserved. In general relativity, it is the object that is called the *Killing vector* that generates the infinitesimal spacetime isometries. Therefore, if one is after a conserved energy definition in general relativity, one can start the investigation with a *timelike* Killing vector, if the spacetime possesses one.

In fact that, this is how one obtains the Komar mass [11] which can be defined via a 3-dimensional volume integral of a special combination of the matter stress-energy tensor, observer 4-velocity and the timelike Killing vector that the spacetime may possess [12]. Therefore, the Komar mass is defined for a *finite* domain of a *stationary spacetime only*. If in addition the spacetime is asymptotic flat, there exist two well-constructed, well-known, *global* energy definitions in general relativity: the Arnowitt-Deser-Misner (ADM) mass-energy [13] and the Bondi mass-energy [14]. The former, ADM mass-energy definition, requires the flatness of 3-spaces at spatial infinity, i.e.,

[1] See [8, 9, 10] for recent, detailed reviews.

when the observers are located at $r \to \infty$.[2] Similarly Bondi's mass-energy definition is obtained via the asymptotic symmetries of the spacetime at null infinity.

On the other hand, in general relativity, we do not only consider isolated objects, i.e., asymptotically flat spacetimes. We would like to understand systems in dynamical spacetimes as well. Moreover, we want to investigate the strong field regime rather than those regions where gravity behaves according to certain fall-off conditions.[3] Accordingly, more concrete mass-energy definitions have been introduced in literature since the 1990s. Among them, the Brown-York [24], Kijowski [25], Epp [26] and Liu-Yau [27] energies are some of the ones which are constructed on or can be linked to a canonical Hamiltonian formalism. These energy definitions are made for finite domains of spacetime, i.e., they are *quasilocal* constructions. Also they do not require any specific spacetime symmetry and therefore applicable for more generic spacetimes. We will refer to the quasilocal energy definitions of Brown and York, Kijowski, Epp and Liu and Yau many times throughout this thesis.

Our current work grew out of wanting to understand fundamental questions about the nature of quasilocal energy in cosmology. According to the supervisor of this thesis, David L. Wiltshire, the dark energy problem we have today is mainly a result of our misinterpretation of the quasilocal energy differences of finite regions in the universe that have different average Ricci curvature.[4] When this PhD study started, he was after a rigorous formalism that relates energy-like quantities defined on small scales to those of statistical averages in cosmology. Therefore the initial goal of this research was to: i) identify the quasilocal kinetic energy corresponding to the expansion of small scale regions in the universe , ii) find the most relevant averaging method associated with it in order to understand the global effects on the large scale.

[2] We will discuss the ADM Hamiltonian formalism in more detail in the next chapter.

[3] In relativistic astrophysics, for example, black holes provide excellent laboratories for understanding some of the important physical mechanisms in the universe. However, the very definition of a black hole is teleological as the entire information about the spacetime has to be known for its existence [15]. In addition to that, in the actual universe, black holes are not just isolated massive stars that have collapsed long time ago and do not interact with their surroundings any longer. On the contrary, in most of the numerical relativity simulations, we investigate the results of their collisions, their behaviour with their binary companions or their accretion mechanisms. Accordingly, our measurements should be considered during a period of *time evolution* of the system. To accommodate this, the *dynamical horizon* concept has become very popular over the last two decades. Dynamical horizons are quasilocal constructions and their definition is not as demanding as the one of an event horizon. Some of the main contributors of this field are Hayward [16, 17, 18, 19, 20] and Ashtekar and Krishnan [21, 22] for $2+2$ and $3+1$ formulations respectively. One can see [23] for a review and for some of the other examples in this field.

[4] See [28, 29, 30, 31] for Wiltshire's Timescape cosmology and his conceptual ideas.

However, for anyone who starts working on energy definitions in a gravitational theory, it is immediately obvious that this problem is like a ball of string with many ends. The identification of concepts such as kinetic and potential mass-energies for a finite region of spacetime is immensely difficult for the generic case due to the non-linear, coupled nature of gravity. Even to start such an investigation, one needs to fully understand how to best define ensemble and time averages of microstates of gravity. This prompts one to look for a statistical theory of gravity and the corresponding equilibrium states. Those equilibrium states are then expected to fit in a thermodynamic picture which is usually applied to only horizons in general relativity.

After the realization of the fact that we were actually trying to bite off more than we could chew, we focused on broader questions about quasilocal energy in general relativity. The main consideration was then how to derive a rigorous method which would allow us to identify at least some of the concepts that we mentioned above. Therefore, in this thesis, we basically present a purely geometric method which allows us to achieve this by consistently defining a system and the corresponding mass-energies it possesses.

The outline of the thesis is as follows. In Chapter 2 we present various formal mathematical preliminaries. This will help for familiarization with the Hamiltonian formulation of field theory and energy definitions in GR. Also we present the mathematical foundations of our approach, which follow from Capovilla and Guven's purely geometric construction [32]. Therefore Chapter 2 is the one in which *energy* and *geometry* come into play.

The definition of a gravitating *system* is crucial for Chapter 3 in which we introduce thermodynamic concepts.[5] We consider systems only defined in spherically symmetric spacetimes which are at thermodynamic equilibrium with their surroundings. This follows from our geometrically defined equilibrium condition. We define certain thermodynamic potentials and present an associated first law. Also, whether or not the thermodynamic equilibrium coincides with the hydrodynamic equilibrium is investigated. This leads us to a virial relation which accounts for both the matter and gravitational energy content of a finite region in question.

However, what we really mean by a system will be only clear in Chapter 4 in which we investigate the finite regions of more generic spacetimes that are not in equilib-

[5] This chapter was published as an article in *Class. Quantum Grav.* 32:165011 [33], arXiv:1506.05801.

rium with their surroundings.[6] We present a more mathematically concrete approach by applying a translation of our formalism from that of Capovilla and Guven [32] to that of Newman and Penrose [34]. We discuss the conditions that one needs to impose on the null cone of the observers, who define those finite regions, in order to end up with well defined energy-like quantities. Following this, we define energy-like quantities which can potentially be exchanged with the surroundings, and a resulting work-energy relation.

In terms of the applications, we consider Schwarzschild, Friedmann-Lemaître-Robertson-Walker and Lemaître-Tolman geometries in Chapter 3. We investigate the effects of time dependence and matter field inhomogeneities on the equilibrium point and the corresponding thermodynamic potentials. In Chapter 4 we apply our construction to a radiating Vaidya geometry, a C-metric and the interior of a Lanczos-van Stockum dust metric. We obtain certain results which might initially seem counter intuitive in a Newtonian framework. However, once we explain the physical content behind these results, they will become more clear.

Although there are various studies in terms of both the energy definitions and the geometric approaches of GR in the literature, this thesis has essentially emerged after careful reading and blending of the ideas presented in three papers: Kijowski's [25], Epp's [26] and Capovilla and Guven's [32]. Kijowski's work presents a big picture of the Hamiltonian formulation of GR in terms of its foundational and mathematical aspects. Epp's work presents a very intuitive investigation in terms of understanding energy and angular momentum by geometric means. This lead us to look for a more concrete geometric construction in the $2+2$ picture that would allow one to link the geometry of 2-surfaces to the observables of finite regions. Such a geometric construction, for arbitrary dimensions, is provided in the work of Capovilla and Guven. We will refer to these papers many times throughout the thesis, as they provide the conceptual and mathematical basis for our own ideas. The reader is advised to refer to them whenever the details we provide here are not clear enough.

Note that we will use natural units in which $c, G, \hbar, k_B$ are taken to be 1 throughout the thesis. The metric signature will be $(-,+,+,+)$.

[6]This chapter has been submitted for publication and can be found in arXiv:1602.07861.

2 Preliminaries

In this chapter, we would like to summarize the investigations and mathematical constructions that will be relevant throughout this thesis. We will provide two main sections: i) Hamiltonian formulations and some of the quasilocal energy definitions in general relativity, ii) A geometrical construction that leads one to the Raychaudhuri equation of a 2-dimensional timelike worldsheet.

Since our investigation is essentially about how to best define the mass-energy of a relativistic system, understanding the necessity behind defining the system's properties *quasilocally* is crucial in order to fully appreciate the original contributions that are given in this thesis.

Therefore, firstly, we will help the reader to get familiarised with the Hamiltonian formulations. We will form analogies between the Hamiltonian formulations of classical mechanics, field theories and general relativity. In order to present an example of *canonical* Hamiltonian formulation of dynamics of the gravitating systems, we will outline Kijowski's approach [25, 35]. Following this, other quasilocal energy definitions – that are either derived via or related to the Hamilton-Jacobi formulation – will be summarized.

Secondly, we will give a short summary of Capovilla and Guven's geometric construction presented in [32]. The extrinsic variables of a timelike worldsheet $\mathbb{T}$, and a spacelike surface $\mathbb{S}$, orthogonal to it at every spacetime point will be introduced. Later, we will present a derivation of Capovilla and Guven's generalized Raychaudhuri equation while also providing the steps one should follow to derive the Raychaudhuri equation of a worldline in a standard way.

At first, the connection between those two sections will probably not be clear to the reader. However, once we start defining the thermodynamic potentials in Chapter

3 and define a work-energy relation in Chapter 4, those connections will be more transparent.

2.1 Hamiltonian formulations and quasilocal energy definitions in the literature

The Hamiltonian formulation of general relativity can be obtained by taking the variation of the action with respect to the spacetime metric, $g_{\mu\nu}$. The metric $g_{\mu\nu}$, i.e., the gravitational field, acts as a source for gravitational energy while at the same time defining the rulers and clocks of the observers who would like to measure and quantify the gravitational energy in question. Note that there is no additional background via which the dynamics of $g_{\mu\nu}$ can be investigated.

Moreover, according to the strong equivalence principle, one can always find a local frame in which $g_{\mu\nu}$ reduces into the metric of flat spacetime. This means that for any gravitational energy definition which is built upon the curvature of the spacetime via $g_{\mu\nu}$ and its first derivatives locally, the gravitational energy is locally zero. Therefore defining the gravitational energy is a challenge for general relativity.

There have been numerous attempts to define the gravitational energy locally via pseudotensors [3, 4, 5, 6, 7]. Construction of those non-tensorial objects requires *a* vector which generates the symmetries of the given spacetime and a covariant derivative operator associated with either a non-dynamical auxilary background metric or a connection. There exist pseudotensors defined by using the partial derivatives of the local coordinates as well. The resultant gravitational energy calculated via the pseudotensor is of course required to be independent of the non-physical background metric or the coordinates. Note that although pseudotensors are helpful for defining the gravitational energy for *preferred* observers in certain situations, most of the time, they cannot succeed in reflecting the coupled matter plus gravitational energy of the system. Moreover, only certain classes of them satisfy the conservation equation with the correct weight [1] [36].

[1] Let $t^{\mu\nu}_{(2k)}$ be a gravitational stress-energy pseudotensor with $k \in \mathbb{R}$. Some of the well known pseudotensors in GR can be defined via $2|g|^{k+1}\left(8\Pi G\, t^{\alpha\beta}_{(2k)} - G^{\alpha\beta}\right) := \partial_\mu \partial_\nu \left(|g|^{k+1}\left[g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\beta\mu}\right]\right)$. Then Einstein field equations imply that $\partial_\alpha\left(|g|^{k+1}\left[t^{\alpha\beta}_{(2k)} + T^{\alpha\beta}\right]\right)$ where $T^{\alpha\beta}$ is the matter stress energy ten-

A careful application of the Hamiltonian approach to the theory of general relativity shows that: i) Part of the Hamiltonian that appears as a 3-dimensional volume integral gives us the constraints, i.e., the first order Einstein field equations. ii) The 2-dimensional boundary Hamiltonian is in fact non-vanishing and it is the part that is physically relevant for the mass-energy measurements of the observers [24, 25]. Because of this, the matter plus gravitational energy of a system is at best defined *quasilocally* in general relativity.

On the other hand, quasilocality is not only demanded for the consistent definition of the matter plus gravitational energy of a relativistic system on account of the equivalence principle. In fact, quasilocality is at the heart of the very idea of measurement. The observables we measure are all obtained by finite size 'laboratories' in finite amounts of time [37]. Also in terms of quantum field theory considerations, physical observables should be associated with finite regions of spacetime [38].

Now that the importance of quasilocal energy definitions is emphasised, we will present more details in terms of the Hamiltonian formulations of general relativity. To do that, we will mostly follow references [25] and [35] of Kijowski in which analogies between Hamiltonian formulations of classical mechanics, field theories and general relativity are formed. This allows one to see the big picture in terms of a canonical Hamiltonian formulation of general relativity. We will also discuss those quasilocal energy definitions which are most relevant to our investigation. Note that a detailed review of quasilocal energy definitions can be found in [36].

2.1.1 Generating functions in classical mechanics, "control" and "response" variables

In order to formulate the dynamics of a system in classical mechanics, consider the function L as a generating function of a $2n$-dimensional Lagrangian submanifold D, in the $4n$-dimensional symplectic space P. In P, let q^i describe the positions, $\dot{q}^i$ velocities, p_i momenta and $\dot{p}_i$ forces acting on particles. Here $\{i = 1...n\}$ and an overdot corresponds to the Newtonian time derivative. Then the Euler-Lagrange equation,

sor. This shows that there is only one pseudotensor, $t^{\mu\nu}_{(-2)}$, which satisfies the conservation of the "total" stress-energy tensor with the correct weight.

i.e.,

$$dL\left(q^i, \dot{q}^i\right) = \dot{p}_i dq^i + p_i d\dot{q}^i \tag{2.1}$$

formulates the dynamical equations for a system. This means that given a system with known particle positions and velocities, once we perturb those values infinitesimally we can get information about the momenta and forces acting on the particles via the mapping $\{\dot{p}_i = \frac{\partial L}{\partial q^i}, p_i = \frac{\partial L}{\partial \dot{q}^i}\}$. For such a case, we have the schematic description of the system as

$$q^i,\ \dot{q}^i \xrightarrow[\dot{p}_i = \frac{\partial L}{\partial q^i},\ p_i = \frac{\partial L}{\partial \dot{q}^i}]{\text{Mapping}} \dot{p}_i,\ p_i$$

Control variables Response variables

Rather than talking about dependent and independent variables, here we introduce Kijowski's language in terms of the "control" and "response" variables in which the former refers to those that can be varied within the system and the later are the ones that result from such variations. The mapping between those two sets of variables are obtained via the relevant partial derivatives of the generating function. Note that in order to model the same dynamics one can pick another generating function in the same symplectic space which satisfies

$$P = [space\, of\, control\, variables] \times [space\, of\, response\, variables] \tag{2.2}$$

Different choices of such a splitting depends on different "control modes" which corresponds to different generating functions.

Now let us apply a Legendre transformation on dL by substituting

$$p_i d\dot{q}^i = d\left(p_i \dot{q}^i\right) - \dot{q}^i dp_i \tag{2.3}$$

in eq. (2.1). Then one can define another generating function, $-H\left(q^i, p_i\right)$ by

$$H = p_i \dot{q}^i - L, \tag{2.4}$$

and

$$-dH\left(q^i, p_i\right) = \dot{p}_i dq^i - \dot{q}^i dp_i \tag{2.5}$$

formulates the same dynamics. In that case we have the following sketch

$$\begin{tikzcd} q^i,\ p_i \arrow[rr, "\text{Mapping}", "\dot{p}_i = -\frac{\partial H}{\partial q^i},\ \dot{q}^i = \frac{\partial H}{\partial p_i}"'] & & \dot{p}_i,\ \dot{q}^i \end{tikzcd}$$

Control variables | Response variables

Note that both of the generating functions, Lagrangian L and Hamiltonian H, keeps the symplectic structure

$$\omega := \left(dp_i \wedge dq^i\right)^{\cdot} = d\dot{p}_i \wedge dq^i + dp_i \wedge d\dot{q}^i \tag{2.6}$$

invariant.

2.1.2 Boundary terms of the Hamiltonian in field theories

In field theory, the information about the dynamics of field, ϕ, is contained in

$$\delta L(\phi, \partial_\nu \phi) = (\partial_\nu p^\nu)\,\delta\phi + p^\nu \delta(\partial_\nu \phi)\,. \tag{2.7}$$

Here the field ϕ plays the role of coordinates q^i in classical mechanics where p^ν are the momenta canonically conjugate to it. The time derivative operator, overdot, is replaced by the partial derivatives with respect to the coordinates x^ν with $\{\nu = 0, 1, 2, 3\}$. Then we have

$$\begin{tikzcd} \phi,\ \partial_\nu\phi \arrow[rr, "\text{Mapping}", "\partial_\nu p^\nu = \frac{\delta L}{\delta\phi},\ p^\nu = \frac{\delta L}{\delta(\partial_\nu\phi)}"'] & & \partial_\nu p^\nu,\ p^\nu \end{tikzcd}$$

Control variables | Response variables

For a Hamiltonian formulation we need some sort of "time" evolution. Therefore let us apply a $(3+1)$ splitting to eq. (2.7) and write

$$\delta L(\phi, \partial_0\phi, \partial_A\phi) = (\pi\delta\phi)^{\cdot} + \partial_A\left(p^A\delta\phi\right), \tag{2.8}$$

with $\{A = 1,2,3\}$. The timelike coordinate is denoted by "0" and $\pi := p^0$. Now we integrate δL over a 3-dimensional spacelike domain V and write

$$\delta\int_V L = \int_V\left(\dot{\pi}\delta\phi + \pi\delta\dot{\phi}\right) + \int_{\partial V} p^{\perp}\delta\phi, \tag{2.9}$$

where the second term on the r.h.s. emerges from Stokes' Theorem since ∂V is the closed boundary of V and $p^{\perp}$ is the component of p^A orthogonal to it. Now apply a Legendre transformation between π and $\dot{\phi}$ by writing

$$\pi\delta\dot{\phi} = \delta\left(\pi\dot{\phi}\right) - \dot{\phi}\delta\pi, \tag{2.10}$$

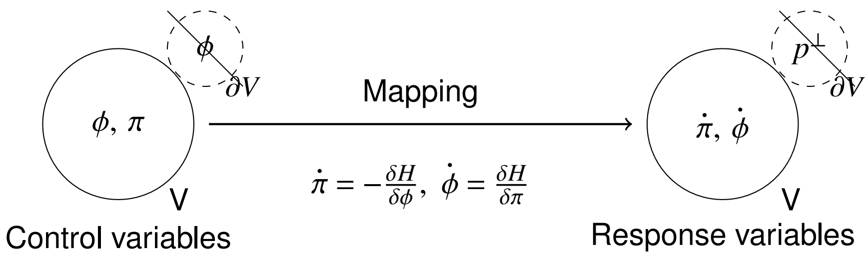
so that one can define a new generating function

$$\mathcal{H} = \int_V H = \int_V\left(\pi\dot{\phi} - L\right) \tag{2.11}$$

that models the same field dynamics by

$$-\delta\mathcal{H} = \int_V\left(\dot{\pi}\delta\phi - \dot{\phi}\delta\pi\right) + \int_{\partial V} p^{\perp}\delta\phi. \tag{2.12}$$

In that case we have,



only if no boundary terms remain after one applies the integration by parts. This is obtained *by hand* when one chooses a Dirichlet type boundary condition, i.e., $\phi|_{\partial V} = constant$. This is typical for field theories. Note that without such a *choice* of boundary condition, the evolution of the system contained in ϕ is influenced by external fields and the system cannot be determined by a Hamiltonian formulation.

In addition to applying a Legendre transformation on the volume integral, one can consider a Legendre transformation on the boundary integral. For example, consider

$$p^{\perp}\delta\phi = \delta\left(p^{\perp}\phi\right) - \phi\,\delta p^{\perp} \tag{2.13}$$

and substitute this into eq. (2.12). Then a new Hamiltonian is defined via

$$\overline{\mathcal{H}} = \mathcal{H} + \int_{\partial V} p^{\perp}\phi, \tag{2.14}$$

and a corresponding generating formula is

$$-\delta\overline{\mathcal{H}} = \int_V \left(\dot{\pi}\delta\phi - \dot{\phi}\delta\pi\right) - \int_{\partial V} \phi\delta p^{\perp} \tag{2.15}$$

which carries information about the same dynamics. However, now, the evolution takes place in a different phase space. In order to have the same Hamiltonian formulation, one has to make the boundary term in eq. (2.15) vanish, as we did in eq. (2.12) for $\mathcal{H}$. This requires imposing Neumann-like boundary conditions, i.e., $p^{\perp}\big|_{\partial V} = constant$.

Note that those boundary conditions are not unique. In a generic case, boundary conditions can effect the field evolution. Most importantly, different boundary conditions correspond to different “insulation” of the physical system.

2.1.3 A canonical Hamiltonian formulation in GR: Kijowski’s approach

In the Lagrangian formulation of GR, the gravitational field strength, $g_{\mu\nu}$, and the spacetime Ricci curvature, R, play the central role. The standard Lagrangian density is given by [39]

$$L = \frac{1}{16\pi}\sqrt{|g|}R, \tag{2.16}$$

where $g = \det g_{\mu\nu}$. In order to construct an action principle, one can either follow the Einstein-Hilbert formulation and consider the variation of L with respect to $g_{\mu\nu}$, or alternatively take variations of L with respect to both $g_{\mu\nu}$ and the connection $\Gamma^{\lambda}{}_{\mu\nu}$ which are treated independently. The latter version is the metric affine formulation of

Palatini [40]. One can also pick a Lagrangian function in which the metric does not appear explicitly and the variation is taken with respect to ${\Gamma^\lambda}_{\mu\nu}$. In that case, $g_{\mu\nu}$ is treated as the momentum canonically conjugate to the connection. This is the purely affine formalism of Kijowski [41].

The Hamiltonian formulation of GR follows from one of the variational principles listed above. Among many Hamiltonian formulations of GR, the Arnowitt-Deser-Misner (ADM) formalism [13] is one of the most widely used. The ADM Hamiltonian, $\mathcal{H}_{ADM}$, follows from an appropriate Legendre transformation of the Einstein-Hilbert Lagrangian density, (2.16).

In order to present $\mathcal{H}_{ADM}$, let us first assume $\left(M, g_{\mu\nu}\right)$ to be a globally hyperbolic spacetime. Then one can pick a global time function, t, such that each $t = constant$ 3-surface is a Cauchy surface. Then the manifold M can be foliated with these Cauchy hypersurfaces, Σ, such that the topology of M is $\mathbb{R}\times\Sigma$. Let us denote **u** to be the timelike unit vector field normal to Σ at each spacetime point with components

$$u^\mu = \left(\frac{1}{N}, -\frac{N^A}{N}\right), \tag{2.17}$$

where N is the lapse function and N^A is the shift vector with $\{A = 1,2,3\}$. Then one decomposes the spacetime metric as the following

$$g_{00} = -N^2 + h_{AB}N^A N^B, \qquad g_{0A} = N_A, \qquad g_{AB} = h_{AB}, \tag{2.18}$$

in which h_{AB} is the 3-metric induced on Σ. The corresponding extrinsic curvature of Σ is given by

$$K_{AB} = -{h^C}_A {h^D}_B D_C u_D \tag{2.19}$$

and D_μ is the spacetime covariant derivative. Let us define an object by

$$P^{AB} = \sqrt{h}\left(K h^{AB} - K^{AB}\right), \tag{2.20}$$

where $h = \det h_{CD}$, $K = K^{AB}h_{AB}$ is the trace of the extrinsic curvature of Σ. Here h_{AB} plays the role of q^i in classical mechanics (or ϕ in field theories) and P^{AB} is the geometrically defined canonical momentum conjugate to h_{AB}.

The ADM Hamiltonian, $\mathcal{H}_{ADM}$, follows from applying a Legendre transformation to L

and integrating the result on Σ, i.e.,

$$\mathcal{H}_{ADM} = \int_{\Sigma}\left(P^{AB}\dot{h}_{AB} - L\right). \tag{2.21}$$

Now consider splitting the terms that appear in the Lagrangian (2.16) into their $(3+1)$ components. By substituting those terms back into eq. (2.21), one may write $\mathcal{H}_{ADM}$ as a functional of the canonical momentum, lapse and shift by

$$\mathcal{H}_{ADM} = \frac{1}{16\pi}\int_{\Sigma}\left(NH + N_A H^A\right), \tag{2.22}$$

where H is the *quadratic constraint* and it is given by

$$H = \frac{1}{\sqrt{h}}\left(P_{AB}P^{AB} - \frac{1}{2}P^2 - \mathcal{R}\sqrt{h}\right), \tag{2.23}$$

and H^A is the *momentum contraint* that is given as

$$H^A = -2D_B P^{AB}. \tag{2.24}$$

Here $P^2 = h_{AB}P^{AB}$ and $\mathcal{R}$ is the Ricci scalar defined with respect to the induced 3-metric. Note that no boundary integral appears in eq. (2.22). Also recall that in eqs. (2.12) and (2.15), which correspond to different Hamiltonians for the same field theory, the boundary integrals were set to zero only by *imposing* Dirichlet and Neumann type boundary conditions. For a general case in general relativity, the total Hamiltonian of any canonical approach involves those boundary terms which do not necessarily vanish. In the ADM formalism, one sets the boundary terms to zero by imposing an asymptotic flatness condition on the spacetime metric. Therefore, $\mathcal{H}_{ADM}$ cannot be considered as the Hamiltonian of a generic system in general relativity.

Note that the symplectic structure of ADM, ω_{ADM}, in the space of initial data $\left(P^{AB}, h_{AB}\right)$ is, accordingly, given by only a volume integral

$$\omega_{ADM} = \frac{1}{16\pi}\int_{\Sigma}\left(dP^{AB}\wedge dh_{AB}\right). \tag{2.25}$$

In his canonical formalism [25], Kijowski considers the same Lagrangian density, (2.16), defined for vacuum[2]. To construct an action principle, the total variation of

[2]However, inclusion of matter fields does not change any of the results that will be presented here.

L is given as

$$\delta L = \delta\left(\frac{1}{16\pi}\sqrt{|g|}\,g^{\mu\nu}R_{\mu\nu}\right) = \underbrace{-\frac{1}{16\pi}\sqrt{|g|}\,G^{\mu\nu}\delta g_{\mu\nu}}_{[\delta L]_1} + \underbrace{\frac{1}{16\pi}\sqrt{|g|}\,g^{\mu\nu}\delta R_{\mu\nu}}_{[\delta L]_2}, \tag{2.26}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor. Thus the entire information of the Einstein-Lagrange equations, $[\delta L]_1 = 0$, should be given in $[\delta L]_2 = 0$ in order for $\delta L = 0$ to hold. From this point on, Kijowski [25] focuses on the $[\delta L]_2$ term only and shows that once $[\delta L]_2$ is integrated over a finite 3-dimensional space, one obtains those boundary terms that should appear in the Hamiltonian formulation of GR and which are missing in the original ADM formalism.[3] This is achieved via writing $[\delta L]_2$ as a total divergence according to

$$[\delta L]_2 = \frac{1}{16\pi}\sqrt{|g|}\,g^{\mu\nu}\delta R_{\mu\nu} = \frac{1}{16\pi}\partial_\lambda\left(\sqrt{|g|}\,g^{\mu\nu}\delta A^\lambda{}_{\mu\nu}\right) \tag{2.27}$$

with

$$A^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\mu\nu} - \delta^\lambda{}_{(\mu}\Gamma^\kappa{}_{\nu)\kappa}, \tag{2.28}$$

which is written purely in terms of the connection. In that case, $A^\lambda{}_{\mu\nu}$ plays the role of the "field" and the weighted inverse metric, $\frac{1}{16\pi}\sqrt{|g|}g^{\mu\nu}$, of spacetime acts as its canonically conjugate momentum according to the analogy.

Now let us, once again, consider a $(3+1)$ splitting of spacetime and integrate $[\delta L]_2$ over the Cauchy surface Σ in adapted coordinates. Then

$$\int_\Sigma [\delta L]_2 = \frac{1}{16\pi}\int_\Sigma\left(\sqrt{|g|}\,g^{\mu\nu}\delta A^0{}_{\mu\nu}\right)^{\cdot} + \frac{1}{16\pi}\int_{\mathbb{S}}\sqrt{|g|}\,g^{\mu\nu}\delta A^1{}_{\mu\nu}. \tag{2.29}$$

Here the overdot refers to the derivative with respect to the time function of the foliation, $\mathbb{S}$ is the closed spacelike boundary of Σ and the index "1" refers to the spacelike adapted coordinate x^1 which is constant on $\mathbb{S}$. Once again, the second term on the r.h.s. of eq. (2.29) follows from Stokes' Theorem. The final form of eq. (2.29) is obtained by Kijowski as the following,

$$\int_\Sigma [\delta L]_2 = -\frac{1}{16\pi}\int_\Sigma\left(h_{AB}\,\delta P^{AB}\right)^{\cdot} + \frac{1}{8\pi}\int_{\mathbb{S}}(\lambda\,\delta\alpha)^{\cdot} - \frac{1}{16\pi}\int_{\mathbb{S}}\gamma_{xy}\,\delta\Pi^{xy}, \tag{2.30}$$

[3] The fact that manifolds with boundary require a remedy was first realised by York [42]. Later Gibbons and Hawking [43] added a boundary action to the Einstein-Hilbert action as a solution. This action is now known as the Gibbons-Hawking-York action whose variation results in the same dynamics as the one of the ADM formalism, once one fixes the metric on the spatial 3-slices.

where $\{x, y = 0, 2, 3\}$. In order to introduce the new terms that appear in eq. (2.30), let us consider a 3-dimensional worldtube, B, which is spanned by $\mathbb{S}$ during its time evolution.

To study the geometry of B, Grabowska and Kijowski introduce four unit vectors [35],
i) **u** is the timelike unit vector orthogonal to Σ,
ii) **ũ** is the timelike unit vector tangent to B,
iii) **n** is the spacelike unit vector tangent to Σ,
iv) **ñ** is the spacelike unit vector orthogonal to **ũ**.
The configuration of these vectors is sketched in Fig. 2.1 and it shows that, the most generic foliation is considered in [35]. In other words, in general, **u** ≠ **ũ** and **n** ≠ **ñ**. In

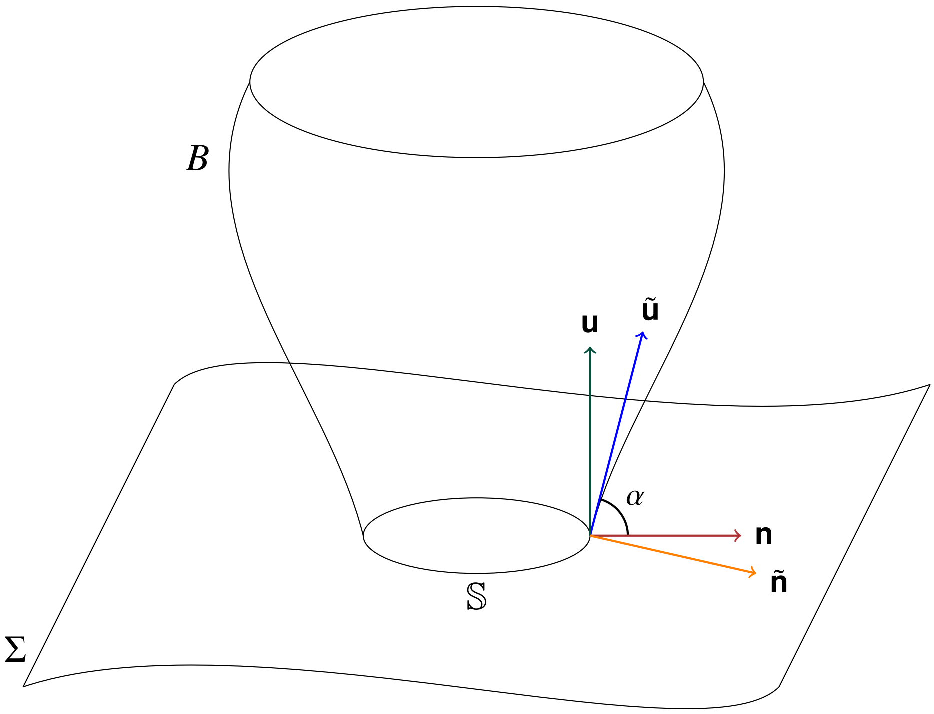


Figure 2.1: Configuration of unit vectors defined with respect to the worldtube B and Cauchy surface Σ. Spacelike 2-surface $\mathbb{S}$ is taken as either the boundary of B or boundary of Σ. Here α is the tilt angle between B and Σ.

the last term of eq. (2.30), the metric induced on B is denoted by γ_{xy}. The associated extrinsic curvature of the worldtube is then

$$\Theta_{xy} = -\gamma^{z}{}_{x}\gamma^{w}{}_{y}D_{z}\tilde{n}_{w}. \tag{2.31}$$

The conjugate momentum of γ_{xy}, analogous to the ADM canonical momentum, (2.20), is defined via

$$\Pi^{xy} = \sqrt{|\gamma|}\,(\Theta\gamma^{xy} - \Theta^{xy}), \tag{2.32}$$

where $\gamma = \det\gamma_{wz}$ and $\Theta = \Theta_{xy}\gamma^{xy}$ is the trace of the extrinsic curvature of B. In the second term, on the r.h.s. of eq. (2.30), α represents the tilt angle between the world-tube B and the Cauchy hypersurface Σ, i.e., $\alpha = \operatorname{arcsinh}(\tilde{\mathbf{u}}|\mathbf{n})$. The term λ represents the density of the induced metric, σ_{ij}, of 2-surface $\mathbb{S}$. It is given by $\lambda = \sqrt{|\det\gamma_{ij}|} = \sqrt{\sigma}$ with $\sigma = \det\sigma_{ij}$ and $\{i, j = 2, 3\}$.

The Hamiltonian description is obtained once one performs a Legendre transformation to eq. (2.30) both on the 3-dimensional volume integral, i.e.,

$$\left(h_{AB}\,\delta P^{AB}\right)^{\cdot} = \dot{h}_{AB}\,\delta P^{AB} + \delta\left(h_{AB}\,\dot{P}^{AB}\right) - \dot{P}^{AB}\,\delta h_{AB}, \tag{2.33}$$

and on the 2-dimensional area integral, i.e.,

$$(\lambda\delta\alpha)^{\cdot} = \dot{\lambda}\delta\alpha + \delta\left(\lambda\dot{\alpha}\right) - \dot{\alpha}\delta\lambda. \tag{2.34}$$

Then with a Hamiltonian function

$$\mathcal{H}_K = -\frac{1}{16\pi}\int_\Sigma\left(h_{AB}\dot{P}^{AB} - L\right) + \frac{1}{8\pi}\int_{\mathbb{S}}\lambda\dot{\alpha}, \tag{2.35}$$

one obtains the generating formula

$$-\delta\mathcal{H}_K = \frac{1}{16\pi}\int_\Sigma\left(\dot{P}^{AB}\,\delta h_{AB} - \dot{h}_{AB}\delta P^{AB}\right) + \frac{1}{8\pi}\int_{\mathbb{S}}\left(\dot{\lambda}\delta\alpha - \dot{\alpha}\delta\lambda\right) - \frac{1}{16\pi}\int_{\mathbb{S}}\left(\gamma_{xy}\delta\Pi^{xy}\right). \tag{2.36}$$

Note that the generating formula of ADM corresponds to

$$\delta\mathcal{H}_{ADM} = \frac{1}{16\pi}\int_\Sigma\left(\dot{P}^{AB}\,\delta h_{AB} - \dot{h}_{AB}\,\delta P^{AB}\right) \tag{2.37}$$

and this clearly shows that the symplectic structure (2.25) corresponding to the ADM Hamiltonian has to be modified. In Kijowski's approach the *total* symplectic structure is given by

$$\omega_K = -\frac{1}{16\pi}\int_\Sigma\left(\delta h_{AB}\wedge\delta P^{AB}\right) + \frac{1}{8\pi}\int_{\mathbb{S}}(\delta\lambda\wedge\delta\alpha). \tag{2.38}$$

We observe that in addition to the the volume integral which corresponds to ω_{ADM}, there exists a boundary integral indicating that the phase space should be enlarged.

Note that $\mathcal{H}_K$ can be rewritten by using the ADM constraints as

$$\mathcal{H}_K = \frac{1}{8\pi}\int_\Sigma \Big(N H + N_A H^A\Big) + \frac{1}{16\pi}\int_{\mathbb{S}}\Big(\Pi^{ij}\gamma_{ij} - \Pi^{00}\gamma_{00}\Big) \tag{2.39}$$

which becomes a pure boundary integral, once the quadratic and momentum constraints $\{H = 0, H^A = 0\}$ are imposed, i.e.,

$$\mathcal{H}_K = \frac{1}{16\pi}\int_{\mathbb{S}}\Big(\Pi^{ij}\gamma_{ij} - \Pi^{00}\gamma_{00}\Big). \tag{2.40}$$

Equation (2.40) can further be simplified and written as a functional of the extrinsic variables of the spacelike 2-surface $\mathbb{S}$, once the $(2+1)$ decomposition of the worldtube B is considered.

For this, let the time evolution vector of B be split into parts that are orthogonal and tangent to $\mathbb{S}$, i.e.,

$$\frac{\partial}{\partial\chi^0} = -s\,\tilde{\mathbf{u}} + s^i\frac{\partial}{\partial\chi^i} \tag{2.41}$$

where s is the lapse function of B and s^i is the shift vector, so that we have

$$\gamma_{00} = -s^2 + s^i s_i \tag{2.42}$$

analogously to the decomposition of g_{00} given in eq. (2.18). Likewise, Π^{xy} is decomposed accordingly and the integrand of eq. (2.40) is found to be

$$\Pi^{ij}\gamma_{ij} - \Pi^{00}\gamma_{00} = \Big(\Pi_{kl}\sigma^{ki}\sigma^{lj}\Big)\gamma_{ij} - 2\Pi^0{}_i\, s^i + s^2\,\Pi^{00}. \tag{2.43}$$

Now let us denote $\Pi^{ij}_{\perp} := \Pi_{kl}\sigma^{ki}\sigma^{lj}$. Under the $(2+1)$ splitting, the generating formula (2.36) follows as

$$\begin{aligned} -\delta\mathcal{H}_K &= \frac{1}{16\pi}\int_\Sigma\Big(\dot{P}^{AB}\delta h_{AB} - \dot{h}_{AB}\delta P^{AB}\Big) + \frac{1}{8\pi}\int_{\mathbb{S}}\Big(\dot{\lambda}\,\delta\alpha - \dot{\alpha}\,\delta\lambda\Big) \\ &- \frac{1}{16\pi}\int_{\mathbb{S}}\Big(\gamma_{00}\,\delta\Pi^{00} + 2\gamma_{0i}\,\delta\Pi^{0i} + \gamma_{ij}\,\delta\Pi^{ij}_{\perp}\Big). \end{aligned} \tag{2.44}$$

Case 1

This is the point where Kijowski starts discussing about the relevant "control" and "response" variables of the boundary Hamiltonian of general relativity. He applies further Legendre transformations between γ_{ij} and $\Pi_{\perp}^{ij}$ and writes another formula that gives the dynamics of spacetime by

$$\begin{aligned} -\delta\mathcal{H}_{K1} &= \frac{1}{16\pi}\int_{\Sigma}\left(\dot{P}^{AB}\delta h_{AB} - \dot{h}_{AB}\delta P^{AB}\right) + \frac{1}{8\pi}\int_{\mathbb{S}}\left(\dot{\lambda}\delta\alpha - \dot{\alpha}\delta\lambda\right) \\ &- \frac{1}{16\pi}\int_{\mathbb{S}}\left(\gamma_{00}\delta\Pi^{00} + 2\gamma_{0i}\delta\Pi^{0i} - \Pi_{\perp}^{ij}\delta\gamma_{ij}\right). \end{aligned} \tag{2.45}$$

This corresponds to a new boundary Hamiltonian which is written as

$$\mathcal{H}_{K1} = -\frac{1}{16\pi}\int_{\mathbb{S}}\left(\Pi^{00}\gamma_{00}\right) - E_0, \tag{2.46}$$

where E_0 is a constant. Note that we are always allowed to add an arbitrary constant to any generating function. This is because it is the generating formula, $-\delta\mathcal{H}_{K1}$, but not the generating function that models the dynamics of the system. Physically this E_0 can be understood as a “reference energy” that provides one with a “datum”. It is only after we specify such a datum that the measured value of the energy of the system makes physical sense.

The $(2+1)$ splitting of the worldtube B, allows eq. (2.46) to be written in terms of the objects that are related to the extrinsic geometry of the spacelike boundary $\mathbb{S}$. Under certain a type of control, which we will discuss in a few paragraphs, $\mathcal{H}_{K1}$ takes its simplest form as

$$\mathcal{H}_{K1} = -\frac{1}{16\pi}\int_{\mathbb{S}}\lambda\frac{\left(k^2 - l^2 - k_0^2\right)}{k_0}. \tag{2.47}$$

Here k is the trace of the extrinsic curvature of $\mathbb{S}$ when embedded into Σ and l is the trace of the extrinsic curvature of $\mathbb{S}$ when embedded into B, which are respectively given as

$$k := \sigma^{ij}k_{ij} = \sigma^{ij}\left[\sigma^{k}{}_{i}\sigma^{l}{}_{j}D_k\tilde{n}_l\right], \tag{2.48}$$

$$l := \sigma^{ij}l_{ij} = \sigma^{ij}\left[\sigma^{k}{}_{i}\sigma^{l}{}_{j}D_k\tilde{u}_l\right], \tag{2.49}$$

and the term k_0 is the extrinsic curvature scalar of a hypothetical $\mathbb{S}$ embedded into a spacelike hypersurface Σ of Minkowski. Then it is easy to observe that $\mathcal{H}_{K1}$ becomes zero for Minkowski space.[4] This makes sense since in a spacetime with no matter and curvature one would expect to measure zero quasilocal energy.

[4]Note that for Minkowski, $l = 0$ automatically.

The simplification of $\mathcal{H}_{K1}$ in (2.46) can be made by means of the geometric identities

$$\begin{aligned} s\Pi^{00} &= -\lambda(k\cosh\alpha + l\sinh\alpha), \\ -\frac{1}{s}\left[\dot{\lambda} - \partial_i\left(\lambda s^i\right)\right] &= -\lambda(k\sinh\alpha + l\cosh\alpha), \\ \Pi^0{}_i + P^1{}_i &= -\lambda\partial_i\alpha. \end{aligned} \tag{2.50}$$

which follow from the definition and the $2+1$ decomposition of the canonical momentum Π^{xy} defined on B. This clearly shows that by controlling Π^{00}, $\Pi^0{}_i$ and $\gamma_{ij} = \sigma_{ij}$ as is done in eq. (2.45), one actually controls the values of the tilt angle, α, the lapse, s, and shift vector components, s^i. Then by choosing

$$\begin{aligned} \Pi^{00} &= -\lambda k_0, \\ \Pi^{0i} &= 0, \\ \dot{\lambda} &= \left(\sqrt{|\det\gamma_{ij}|}\right)^{\bullet} = 0. \end{aligned}$$

and making use of the equation set (2.50), the equation (2.46) simplifies into eq. (2.47) on account of $s^2 = \left(k^2 - l^2\right)/k_0^2$. Note that in this approach, k_0 *emerges* due to the controls Kijowski imposes on the system to exclude the effects of boosts, rotations and translations in $\mathcal{H}_{K1}$ which is achieved by assigning flat values to Π^{00} and Π^{0i}.

Case 2

Alternatively, Kijowski applies another Legendre transformation between the terms $\{s^2, \Pi^{00}\}$ and $\{s^i, \Pi^0{}_i\}$ in the boundary integral of eq. (2.45). Then one has a new generating formula

$$\begin{aligned} -\delta\mathcal{H}_{K2} &= \frac{1}{16\pi}\int_\Sigma\left(\dot{P}^{AB}\,\delta h_{AB} - \dot{h}_{AB}\,\delta P^{AB}\right) + \frac{1}{8\pi}\int_{\mathbb{S}}\left(\dot{\lambda}\delta\alpha - \dot{\alpha}\delta\lambda\right) \\ &+ \frac{1}{16\pi}\int_{\mathbb{S}}\left(\Pi^{00}\,\delta\left(-s^2\right) + 2\Pi^0{}_i\,\delta s^i + \Pi^{ij}_{\perp}\,\delta\gamma_{ij}\right). \end{aligned} \tag{2.51}$$

with the corresponding Hamiltonian

$$\mathcal{H}_{K2} = -\frac{1}{8\pi}\int_{\mathbb{S}}\left(\Pi^{00}\,\gamma_{00} + \Pi^{0i}\,\gamma_{0i}\right) - E_0. \tag{2.52}$$

From eq. (2.51) we observe that the control variables are now $\{s, s^i, \gamma_{ij}\}$ which include the entire information encoded in the worldtube metric γ_{xy}. Therefore, by imposing certain values on the new control variables, one fixes the metric on B.

After substituting the following choice of control variables

$$s^2 = 1, \tag{2.53}$$

$$s^i = 0, \tag{2.54}$$

$$\dot{\lambda} = 0, \tag{2.55}$$

into the equation set (2.50), then solving for Π^{00}, α and Π^{0i}, one can write the Hamiltonian (2.52) in terms of the extrinsic curvature scalars of $\mathbb{S}$ by

$$\mathcal{H}_{K2} = -\frac{1}{8\pi}\int_{\mathbb{S}} \lambda\left(\sqrt{k^2 - l^2} - k_0\right) \tag{2.56}$$

From now on we will denote $\mathcal{H}_{K1}$ and $\mathcal{H}_{K2}$ as E_{K1} and E_{K2} respectively, to indicate that they do refer to *energy* rather than any other quantity. We will discuss their physical interpretations together with other quasilocal energy definitions in the literature in the next few subsections. Note that we do not follow a chronological order in terms of the introduction of those definitions to the literature.

2.1.4 Brown-York (BY) Energy

Brown and York [24] also followed a Hamilton-Jacobi approach to define a quasilocal energy on a 2-dimensional spacelike boundary, $\mathbb{S}$, of the worldtube B.

Hamiltonian formulation of BY is fundamentally different than the ones of ADM or Kijowski in the sense that the starting point of the action principle is not the Einstein-Hilbert Lagrangian, (2.16). Rather, they consider an action, I, by adding extra boundary terms to the standard Einstein-Hilbert action by

$$I = \frac{1}{16\pi}\int_M \sqrt{|g|}R + \frac{1}{8\pi}\int_\Sigma \sqrt{h}K - \frac{1}{8\pi}\int_B \sqrt{|\gamma|}\Theta, \tag{2.57}$$

and write its variation as

$$\delta I = \frac{1}{16\pi}\int_M \sqrt{|g|}\,G_{\mu\nu}\,\delta g^{\mu\nu} + \frac{1}{8\pi}\int_{\Sigma_{t_0}}^{\Sigma_t} \sqrt{h}\,P^{AB}\,\delta h_{AB} - \frac{1}{8\pi}\int_B \sqrt{|\gamma|}\,\Pi^{xy}\,\delta\gamma_{xy}. \tag{2.58}$$

Then, when Einstein field equations are satisfied, one has to fix h_{AB} on Σ_{t_0} and Σ_t also fix γ_{xy} on B in order to have $\delta I = 0$. These give the boundary conditions/"controls" of the BY formalism.

According to this approach, the so-called *gravitational* stress-energy tensor, $T_B^{\mu\nu}$, defined on B captures the coupled effects of matter and gravitation[5]. It is defined via the canonical momentum of γ_{xy} by

$$T_B^{xy} = \frac{2}{\sqrt{|\gamma|}}\Pi^{xy} = \frac{2}{\sqrt{|\gamma|}}\frac{\delta I}{\delta\gamma_{xy}}. \tag{2.59}$$

When one projects this stress-energy tensor tangentially and normally to the spacelike 2-boundary of B, one obtains the quasilocal energy, momentum and spatial stress densities. The Brown-York quasilocal energy density (energy per 2-surface area) is then written as

$$\varepsilon = u_x u_y T_B^{xy}. \tag{2.60}$$

After one splits T_B^{xy} into its $(2+1)$ components, the quasilocal energy corresponding to ε reads as [24],

$$E_{\mathrm{BY}} = -\frac{1}{8\pi}\int_{\mathbb{S}} \sqrt{\sigma}\,(k - k_0) \tag{2.61}$$

where k is again the extrinsic curvature scalar of $\mathbb{S}$ when it is embedded in Σ and k_0 is the corresponding extrinsic curvature evaluated for a suitable reference spacetime. Brown and York also emphasise the fact that for a positive curvature embedding of $\mathbb{S}$ into flat spaces of the chosen reference spacetime, one is guaranteed a unique embedding.

Note that Kijowski's quasilocal energy E_{K2}, given in eq. (2.56), becomes equal to E_{BY} in a specific guage, i.e., $l = 0$. This is mainly because in both of the definitions, one chooses the same boundary condition by imposing a fixed metric on B. However, E_{K2} has an extra gauge freedom, due to the metric on Σ not being fixed.

[5] In order for $T_B^{\mu\nu}$ to include the effect of matter fields, one simply adds the matter action to I. This does not change any of the results.

2.1.5 Epp's (E) Energy

Epp follows the BY formalism and considers a modification of $E_{\rm BY}$ by forming an analogy between the special relativistic (SR) proper mass of a particle and the "invariant" mass-energy of a system [26]. According to his analogy

$$\sqrt{E^2-p^2} \qquad (SR) \qquad \longrightarrow \sqrt{k^2-l^2} \qquad (GR), \tag{2.62}$$

where E and p are the relativistic energy and momentum of the particle respectively. According to Epp, it is the *mean extrinsic curvature* of $\mathbb{S}$, i.e. $\sqrt{k^2-l^2}$, rather then k that reflects the total mass-energy content within a bounded region. Therefore, in order to define the invariant quasilocal mass energy he considers [26],

$$E_{\rm E}^{\rm Physical} = -\frac{1}{8\pi}\int_{\mathbb{S}} \sqrt{\sigma}\,\sqrt{k^2-l^2}, \tag{2.63}$$

$$E_{\rm E}^{\rm Reference} = -\frac{1}{8\pi}\int_{\mathbb{S}} \sqrt{\sigma}\,\sqrt{k_{ref}^2-l_{ref}^2}, \tag{2.64}$$

so that

$$E_{\rm E} = E_{\rm E}^{\rm Physical} - E_{\rm E}^{\rm Reference}. \tag{2.65}$$

Note that for the case of the reference spacetime being Minkowski, E_E does not exactly reduce to $E_{\rm K2}$ due the definition of mean extrinsic curvature having an extra factor of $1/2$ in Epp's appraoch [26].

2.1.6 Liu-Yau (LY) Energy

In Liu and Yau's work [27] there is no reference to a timelike 3-dimensional boundary. Liu and Yau considered the embedding of $\mathbb{S}$ directly into a spacetime domain D by taking its two normal null vectors and the corresponding mean extrinsic curvature. That provides a well-defined quasilocal energy under the direct embedding of a 2-dimensional spacelike surface into a 4-dimensional spacetime. In fact, this idea is closest to the heart of the method we use in this thesis. When Liu and Yau's work is converted from their original notation into the one used here, their energy expression becomes [36]

$$E_{\rm LY} = -\frac{1}{8\pi}\int_{\mathbb{S}} \sqrt{\sigma}\left[\sqrt{(k^2-l^2)}-k_0\right]. \tag{2.66}$$

The reference energy is obtained by embedding $\mathbb{S}$ into the 3-dimensional Euclidean space, $\mathbb{R}^3$, and calculating its extrinsic curvature, k_0. This isometric embedding is unique up to the isometries of $\mathbb{R}^3$. Note that their quasilocal energy expression is exactly equal to Kijowski's energy given by (2.56). The positivity of the Kijowski-Liu-Yau energy, denoted E_{KLY} from now on, has been proven by Liu and Yau [27, 44]. It is a widely accepted quasilocal energy definition [45, 36].

Important Remark:

The literature is divided into two camps in terms of the definition of the extrinsic scalars k and l. For example, according to the definition given in eq. (2.48), $k_0 = +\frac{2}{r}$ for a round 2-sphere. This notation was used in Epp's [26], Liu and Yau's [27] and in Szabados's review article [36]. On the other hand, Brown and York [24] and Kijowski [25] follow the formal notation for the extrinsic curvature, so that eqs. (2.48) and (2.49) have an extra negative sign. Accordingly $k_0 = -\frac{2}{r}$ for a round 2-sphere in their notation. In this thesis, we follow the notation used by the first camp since the "positivity" theorem was first presented in this notation [27]. Moreover, we suspect most researchers refer to Szabados' review article to compare and contrast various quasilocal energy definitions. Therefore, in Kijowski's and Brown and York's original papers, E_{K1}, E_{K2} and E_{BY} are given in a different form than the one in eqs. (2.47), (2.56) and (2.61) respectively.

2.1.7 Misner-Sharp-Hernandez (MSH) Energy

In Chapter 3 we will focus on systems that are in thermodynamic equilibrium with their surroundings and in horizon thermodynamics there is a broad consensus [18, 46, 47, 48, 49] on the choice of the internal energy of a generic spherically symmetric spacetime. It is usually taken as the Misner-Sharp-Hernandez energy [50, 51] which does not follow from a Hamiltonian approach.

Let us consider a spherically symmetric spacetime metric with coordinates $\{y^\alpha, \theta, \phi\}$ where $\{y^\alpha\} = \{t, r\}$,

$$ds^2 = \Upsilon_{\alpha\beta} dy^\alpha dy^\beta + R^2(y)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.67}$$

R being the areal radius. In order to study time evolution, one can pick a preferred timelike vector, called the Kodama vector [52], which can be used to define surface gravity for dynamic spherically symmetric spacetimes [18]. The surface gravity, up to a constant, is in general related to the temperature defined on the horizon. The Kodama vector is unique and it is parallel to the timelike Killing vector in static spacetimes. Its components are given by,

$$K^\alpha(y) = \epsilon^{\alpha\beta}\partial_\beta R, \qquad K^\theta = 0, \qquad K^\phi = 0, \tag{2.68}$$

where $\epsilon_{\alpha\beta}$ is the Levi-Civita tensor in 2-dimensions. Now consider a 3-dimensional spacelike hypersurface, Σ, with induced metric, h_{AB}, and unit normal, n^ν, aligned with the Kodama vector. The Kodama vector is associated with conserved charges including an energy

$$E_{\mathrm{MSH}} = \int_\Sigma \sqrt{h}\, T_{\mu\nu} K^\mu n^\nu, \tag{2.69}$$

where $T_{\mu\nu}$ is the stress-energy tensor of matter in the 4-dimensional spacetime. This defines the Misner-Sharp-Hernandez energy which can also be written

$$E_{\mathrm{MSH}} = \frac{R}{2}\left(1 - \Upsilon^{\alpha\beta}\partial_\alpha R \partial_\beta R\right). \tag{2.70}$$

2.1.8 On radial boost invariance

The quasilocal energies E_{K1}, E_{E} and E_{KLY} of a system are obtained via the mean extrinsic curvature of a 2-dimensional spacelike boundary, which we will call the *screen*. For a spherically symmetric spacetime, for example, the generator of the quasilocal energy is not just a single timelike vector. Rather, one needs to consider both the future pointing timelike normal and the outward pointing radial spacelike normal in order to calculate the mean extrinsic curvature of the screen.

Recall that following the ideas of Epp [26] E_{KLY} can be interpreted as a *proper* mass-energy of the system. It is known that both E_{KLY} and E_{K1} are invariant under radial boosts of the quasilocal observers who define the 2-surface [25, 36, 26, 27]. Such a property is necessary to define a system consistently since one needs the ability to keep constant the degrees of freedom that *define* the screen enclosing the system. In the case of spherical symmetry, $\{\theta, \phi\}$ are the coordinates on the screen that are kept constant. Then the evolution of the system is investigated by perturbing the screen along the remaining degrees of freedom, parametrized by the $\{t, r\}$ coordinates. Thus the screen observers agree on the quasilocal energy content of the *same* system irrespective of them being boosted or having instantaneous radial accelerations with respect to any other screen.

We note that in this picture, the Kodama vector is an object that resides on the temporal-radial plane. The fact that the Kodama vector is associated with a conserved Misner-Sharp-Hernandez energy has previously been described as a *miracle* [53]. Here we emphasize that since E_{K1} matches E_{MSH} under spherical symmetry, this "miracle" is a natural consequence of any consistent quasilocal Hamiltonian formalism of general relativity. Thus one does not need to define a single *preferred* observer in the energy calculations.

The distinction of the degrees of freedom that are used to define the system and the ones used in the investigation of its evolution is crucial for the interpretation of the formalism introduced in the next section.

2.2 Raychaudhuri equation and the geometry of a timelike worldsheet

In [32], Capovilla and Guven construct a formalism to investigate the extrinsic geometry of an arbitrary dimensional timelike worldsheet embedded in an arbitrary dimensional spacetime. In Chapter 3 and Chapter 4 we will use their formalism to investigate the properties of a 2-dimensional worldsheet embedded in a 4-dimensional spacetime. In this section, we will present the full details of their geometric construction as applied to the specific case of a $(2+2)$ geometry.

2.2.1 Geometry of the worldsheet

Let us consider an embedding of an oriented worldsheet with an induced metric, η_{ab}, written in terms of orthonormal basis tangent vectors, $\{E_a\}$,

$$g(E_a, E_b) = \eta_{ab}, \tag{2.71}$$

where $g_{\mu\nu}$ is the 4-dimensional spacetime metric. Now consider the two unit normal vectors, $\{N^i\}$, of the worldsheet which are defined up to a local rotation by,

$$g(N_i, N_j) = \delta_{ij} \tag{2.72}$$

$$g(N^i, E_a) = 0, \tag{2.73}$$

where $\{a,b\} = \{\hat{0},\hat{1}\}$ and $\{i,j\} = \{\hat{2},\hat{3}\}$ are the diad indices and the Greek indices refer to 4-dimensional spacetime coordinates. Also note that to raise (or lower) the indices of tangential and normal diad indices one should use η^{ab} (or η_{ab}) and δ^{ij} (or δ_{ij}) respectively. We choose $\eta_{ab} = diag(-1,1)$ and $\delta_{ij} = diag(1,1)$ throughout this thesis.

Capovilla and Guven define three types of covariant derivatives whose distinction we will now introduce. Let the torsionless covariant derivative defined by the spacetime coordinate metric be D_μ and its projection onto the worldsheet be denoted by $D_a = E^\mu{}_a D_\mu$. On the worldsheet $\mathbb{T}$, ∇_a is defined with respect to the intrinsic metric and $\tilde{\nabla}_a$ is defined on tensors under rotations of the normal frame which we call $\mathbb{S}$. Likewise the projection of the spacetime covariant derivative on the instantaneous spacelike 2-surface $\mathbb{S}$ is $D_i = N^\mu{}_i D_\mu$. On $\mathbb{S}$, ∇_i is defined with respect to the intrinsic metric and $\tilde{\nabla}_i$ is defined on tensors under rotations of the tangent frame, $\mathbb{T}$.

To study the deformations of $\mathbb{T}$ and $\mathbb{S}$, the following extrinsic variables are introduced [32]. The extrinsic curvature, Ricci rotation coefficients and extrinsic twist of $\mathbb{T}$ are respectively defined by,

$$K_{ab}{}^i = -g_{\mu\nu}\left(D_a E^\mu{}_b\right)N^{\nu i} = K_{ba}{}^i, \tag{2.74}$$

$$\gamma_{abc} = g_{\mu\nu}\left(D_a E^\mu{}_b\right)E^\nu{}_c = -\gamma_{acb}, \tag{2.75}$$

$$w_a{}^{ij} = g_{\mu\nu}\left(D_a N^{\mu i}\right)N^{\nu j} = -w_a{}^{ji} \tag{2.76}$$

while the extrinsic curvature, Ricci rotation coefficients and extrinsic twist of $\mathbb{S}$ are respectively defined by,

$$J_a{}^{ij} = g_{\mu\nu}\left(D^i E^\mu{}_a\right)N^{\nu j}, \tag{2.77}$$

$$\gamma_{ijk} = g_{\mu\nu}\left(D_i N^\mu{}_j\right)N^\nu{}_k = -\gamma_{ikj}, \tag{2.78}$$

$$S_{ab}{}^{i} = g_{\mu\nu}\left(D^{i}E^{\mu}{}_{a}\right)E^{\nu}{}_{b} = -S_{ba}{}^{i}. \tag{2.79}$$

By using those extrinsic variables one can investigate how the orthonormal basis $\{E_a, N^i\}$ varies when perturbed on $\mathbb{T}$ according to,

$$D_a E_b = \gamma_{ab}{}^{c} E_c - K_{ab}{}^{i} N_i, \tag{2.80}$$

$$D_a N^i = K_{ab}{}^{i} E^b + w_a{}^{ij} N_j, \tag{2.81}$$

or perturbed on $\mathbb{S}$ according to,

$$D_i E_a = S_{abi} E^b + J_{aij} N^j, \tag{2.82}$$

$$D_i N_j = -J_{aij} E^a + \gamma_{ij}{}^{k} N_k. \tag{2.83}$$

For an arbitrary tensor $\Phi^{i_1 \ldots i_n}$ the worldsheet covariant derivative, $\tilde{\nabla}_a$, is then written as

$$\tilde{\nabla}_a \Phi^{i_1 \ldots i_n} = \nabla_a \Phi^{i_1 \ldots i_n} - w_a{}^{i_1 j} \Phi_j{}^{i_2 \ldots i_n} - \ldots - w_a{}^{i_n j} \Phi^{i_1 \ldots i_{n-1}}{}_{j}, \tag{2.84}$$

in which $w_a{}^{ij}$ transforms as a connection with respect to a normal frame rotation and

$$\nabla_a \Phi_b = D_a \Phi_b - \gamma_{abc} \Phi^c \tag{2.85}$$

is the covariant derivative defined via the induced metric on $\mathbb{T}$. Likewise, the normal frame covariant derivative, $\tilde{\nabla}_i$, is defined via

$$\tilde{\nabla}_i \Phi_{a_1 \ldots a_n} = \nabla_i \Phi_{a_1 \ldots a_n} - S_{a_1 b i} \Phi^b{}_{a_2 \ldots a_n} - \ldots - S_{a_n b i} \Phi_{a_1 \ldots a_{n-1}}{}^{b}, \tag{2.86}$$

where $S_{ab}{}^{i}$ transforms as a connection under the tangent frame rotation and

$$\nabla_i \Phi_j = D_i \Phi_j - \gamma_{ijk} \Phi^k \tag{2.87}$$

is the covariant derivative defined via the induced metric on $\mathbb{S}$.

Later in this section we will see that the generalised Raychaudhuri equation can be written in terms of these extrinsic variables and their relevant covariant derivatives.

Therefore, at this point, we will give a break from the CG formalism and remind the reader of more familiar form of the Raychaudhuri equation in the (3+1) formalism.

2.2.2 Raychaudhuri equation of a worldline

For a timelike flow, the gradient of the velocity field can be split into three parts: shear, vorticity and expansion as the following

$$D_\mu u_\nu = \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3} h_{\mu\nu}\Theta. \tag{2.88}$$

Here u^μ is the 4-velocity of observers and

$$h^\mu{}_\nu = g^\mu{}_\nu + u^\mu u_\nu \tag{2.89}$$

is the operator that projects tensors onto the 3-dimensional surfaces locally orthogonal to the flow[6]. The *expansion* is the pure trace part of the divergence and given by

$$\Theta = D_\mu u^\mu. \tag{2.90}$$

The symmetric, traceless part is the *shear* and it is defined as[7]

$$\sigma_{\mu\nu} = \frac{1}{2}\left(D_\mu u_\nu + D_\nu u_\mu\right) - \frac{1}{3} h_{\mu\nu}\Theta. \tag{2.91}$$

The *vorticity* is the antisymmetric traceless part, i.e.,

$$\omega_{\mu\nu} = \frac{1}{2}\left(D_\nu u_\mu - D_\mu u_\nu\right). \tag{2.92}$$

The derivation of the Raychaudhuri equation in a standard way in the $(3+1)$ formalism, can be obtained by first defining a second rank tensor by $B_{\mu\nu} = D_\mu u_\nu$ and evaluating the quantity $u^\alpha D_\alpha B_{\mu\nu}$ [12]. Then once one splits this identity into pure trace, traceless symmetric and traceless antisymmetric parts one obtains the following equations

[6] By the Frobenius Theorem, these spaces form hypersurfaces if only if the vorticity vanishes, $\omega_{\mu\nu} = 0$. We consider the general case with nonvanishing vorticity.

[7] It is unfortunate that $\sigma_{\mu\nu}$ is used to denote both the shear tensor and the induced 2-metric of a closed spacelike boundary in the literature. We hope it is clear to the reader which one we are actually referring to depending on the context.

respectively,

$$\begin{aligned}
\frac{d\Theta}{d\chi} &= -\frac{1}{3}\Theta^2 - \sigma^2 + \omega^2 - R_{\mu\nu}u^\mu u^\nu, && (2.93)\\
\frac{\sigma_{\mu\nu}}{d\chi} &= -\frac{2}{3}\Theta\,\sigma_{\mu\alpha}\sigma^\alpha{}_\nu - \omega_{\mu\alpha}\omega^\alpha{}_\nu + \frac{1}{3}h_{\mu\nu}\left(\sigma^2 - \omega^2\right) + C_{\alpha\nu\mu\beta}u^\alpha u^\beta + \frac{1}{2}\tilde{R}_{\mu\nu}, && (2.94)\\
\frac{d\omega_{\mu\nu}}{d\chi} &= -\frac{2}{3}\Theta\,\omega^\mu{}_\nu - 2\sigma^\alpha{}_{[\nu}\omega_{\mu]\alpha}. && (2.95)
\end{aligned}$$

where χ is an affine parameter on integral curves of u^μ which can be chosen as the proper time for timelike curves, $\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}$, $\omega^2 = \omega_{\mu\nu}\omega^{\mu\nu}$, $C_{\alpha\nu\mu\beta}$ is the Weyl tensor and

$$\tilde{R}_{\mu\nu} = h_{\mu\alpha}h_{\nu\beta}h^{\alpha\beta} - \frac{1}{3}h_{\mu\nu}h_{\alpha\beta}R^{\alpha\beta} \qquad (2.96)$$

with $R_{\alpha\beta}$ being the Ricci tensor. Note that the identities (2.93)-(2.95) are purely geometric relations. The Einstein field equations are not imposed on them. Also, it is usually the trace part, eq. (2.93), that is referred to as the “Raychaudhuri equation” by many researchers even though the traceless identities, (2.94) and (2.95), also carry information about the dynamics of the timelike congruences. For other delicate issues and a recent review of the Raychaudhuri equation one can see, for example, [54].

2.2.3 Raychaudhuri equation of a worldsheet

Now we will present a derivation of the *generalized* Raychaudhuri equation of Capovilla and Guven. In the case of the (3+1) formalism, the timelike vector field $E^\mu{}_a$ that lives on $\mathbb{T}$ can directly be set to u^μ. In that case the Raychaudhuri equation of CG would contain the same information as equations (2.93)-(2.95), providing us with the knowledge of the dynamics of how much a congruence of timelike *worldlines* expands, shears or rotates. However, in section (2.1) we observed that the extrinsic curvature of a closed spacelike 2-surface – when it is perturbed *both* in a timelike and a in spacelike direction – gives information about the quasilocal matter plus gravitational energy of a system that it encloses. Therefore, we will apply the CG formalism to a 2-dimensional worldsheet embedded in 4-dimensional spacetime. Then $E^\mu{}_a$, with $\{a, b\} = \{\hat{0}, \hat{1}\}$, will represent the Lorentzian signature dyad orthogonal to the spacelike 2-surface $\mathbb{S}$. The Raychaudhuri equation constructed from this dyad carries information about how much the congruence of timelike *worldsheets* – rather than worldlines – expands, shears or rotates.

Now analogously to defining the tensor $B_{\mu\nu} = D_\mu u_\nu$, one can define $J_a{}^{ij} = g_{\mu\nu}\left(D^i E^\mu{}_a\right) N^{\nu j}$ which actually corresponds to the extrinsic curvature of $\mathbb{S}$ given in eq. (2.77). Then the derivation of the generalized Raychaudhuri equation can be obtained via $\tilde{\nabla}_b J_a{}^{ij}$ in analogy to the object $u^\alpha D_\alpha\, B_{\mu\nu}$ in the standard approach. Note that here one needs to use the worldsheet covariant derivative $\tilde{\nabla}_b$ in order to find the 'divergence' of $J_a{}^{ij}$, as it is the operator which successfully transforms the tensors on the worldsheet. Then by using eqs. (2.84) and (2.85) we write

$$\tilde{\nabla}_b J_{aij} = \underbrace{D_b J_{aij}}_{1} - \gamma_{ba}{}^c J_{cij} - w_{bi}{}^k J_{akj} - w_{bj}{}^k J_{aik}. \tag{2.97}$$

Let us start with considering the object $D_b J_{aij}$. By using the definition (2.77) and the metric compatibility condition we get

$$1 = \quad D_b J_{aij} = D_b\left[g_{\mu\nu} N^\mu{}_j D_i E^\nu{}_a\right] = \underbrace{g_{\mu\nu}\left(D_b N^\mu{}_j\right)\left(D_i E^\nu{}_a\right)}_{2} + \underbrace{g_{\mu\nu} N^\mu{}_j D_b D_i \left(E^\nu{}_a\right)}_{3}. \tag{2.98}$$

In order to simplify 2 we use eqs. (2.81) and (2.82) and write

$$\begin{aligned} 2 \quad &= \quad g_{\mu\nu}\left(K_{bcj} E^{\mu c} + w_{bjk} N^{\mu k}\right)\left(S_{adi} E^{\nu d} + J_{ail} N^{\nu l}\right) &(2.99)\\ &= \quad g_{\mu\nu} E^{\mu c} E^{\nu d} K_{bcj} S_{adi} + g_{\mu\nu} N^{\mu k} N^{\nu l} w_{bjk} J_{ail} \\ &+ \quad g_{\mu\nu} E^{\mu c} N^{\nu l} K_{bcj} J_{ail} N^{\nu l} + g_{\mu\nu} E^{\nu d} N^{\mu k} w_{bjk} S_{adi}. &(2.100)\end{aligned}$$

Since $g\left(E^c, E^d\right) = \eta^{cd}$, $g\left(N^k, N^l\right) = \delta^{kl}$ and $g\left(E^c, N^k\right) = 0$ we end up with

$$2 = K_b{}^c{}_j S_{aci} + w_{bj}{}^k J_{aik}. \tag{2.101}$$

In order to simplify 3 we will use the Ricci identity[8] for $E^\mu{}_a$, i.e.,

$$D_b D_i E^\nu{}_a - D_i D_b E^\nu{}_a - D_{(D_b N_i - D_i E_b)} E^\nu{}_a = R^\nu{}_{\mu b i} E^\mu{}_a, \tag{2.102}$$

[8]For an arbitrary vector, A^μ, the Ricci identity is given as $D_\alpha D_\beta A^\mu - D_\beta D_\alpha A^\mu - D_{[\mathbf{e}_\alpha, \mathbf{e}_\beta]} A^\mu = R^\mu{}_{\nu\alpha\beta} A^\nu$, where $R_{\mu\nu\alpha\beta}$ is the Riemann tensor of the spacetime and the last term on the l.h.s. vanishes only for coordinate basis vectors, $\mathbf{e}_\alpha$.

and hence

$$D_b D_i E^\nu{}_a = D_i D_b E^\nu{}_a + \left(D_b N_i\right)^\rho D_\rho E^\nu{}_a - \left(D_i E_b\right)^\rho D_\rho E^\nu{}_a + R^\nu{}_{abi}. \tag{2.103}$$

Then using eq. (2.103) we find,

$$\begin{aligned} ③ &= g\left(N_j, D_b D_i E_a\right) \\ &= \underbrace{g\left(N_j, D_i D_b E_a\right)}_{④} + \underbrace{g\left(N_j, \left(D_b N_i\right)^\rho D_\rho E_a - \left(D_i E_b\right)^\rho D_\rho E_a\right)}_{⑤} + \underbrace{g\left(N_j, R\left(E_b, N_i\right) E_a\right)}_{⑥}. \end{aligned} \tag{2.104}$$

where $R(E_b, N_i) E_a = R^\nu{}_{abi}$. By using eq. (2.80) we can write

$$④ = g\left(N_j, D_i D_b E_a\right) = g\left(N_j, D_i\left[\gamma_{ba}{}^c E_c\right]\right) - g\left(N_j, D_i\left[K_{ba}{}^k N_k\right]\right), \tag{2.105}$$

and by further considering eqs. (2.82) and (2.83) we get

$$\begin{aligned} ④ &= g\left(N_j, \gamma_{ba}{}^c\left[S_{cdi} E^d + J_{cil} N^l\right]\right) + g\left(N_j, E_c D_i \gamma_{ab}{}^c\right) \\ &- g\left(N_j, K_{ba}{}^k\left[-J_{dik} E^d + \gamma_{ik}{}^l N_l\right]\right) - g\left(N_j, N_k D_i K_{ba}{}^k\right), \\ &= \gamma_{ba}{}^c J_{cij} - K_{ba}{}^k \gamma_{ikj} - D_i K_{baj}, \end{aligned} \tag{2.106}$$

in which the final equality comes from the fact that $g\left(N^k, N^l\right) = \delta^{kl}$ and $g\left(E^c, N^k\right) = 0$. Simplification of ⑤ is obtained once eqs. (2.81) and (2.82) are considered, i.e.,

$$\begin{aligned} ⑤ &= g\left(N_j, \left(D_b N_i\right)^\rho D_\rho E_a - \left(D_i E_b\right)^\rho D_\rho E_a\right) \\ &= g\left(N_j, \left[K_b{}^c{}_i E^\rho{}_c + w_{bi}{}^k N^\rho{}_k\right] D_\rho E_a\right) - g\left(N_j, \left[S_b{}^c{}_i E^\rho{}_c + J_{bi}{}^k N^\rho{}_k\right] D_\rho E_a\right) \end{aligned} \tag{2.107}$$

Substitution of eqs. (2.80) and (2.82) into eq. (2.107) gives

$$\begin{aligned} ⑤ &= g\left(N_j, \left[K_b{}^c{}_i - S_b{}^c{}_i\right]\left[\gamma_{ca}{}^d E_d - K_{ca}{}^k N_k\right]\right) + g\left(N_j, \left[w_{bi}{}^k - J_{bi}{}^k\right]\left[S_{adk} E^d + J_{ak}{}^l N_l\right]\right), \\ &= -K_{caj} K_b{}^c{}_i + K_{caj} S_b{}^c{}_i + J_{akj} w_{bi}{}^k - J_{akj} J_{bi}{}^k. \end{aligned} \tag{2.108}$$

Now recall that what we ultimately want to derive is $\tilde{\nabla}_b J_{aij}$, i.e., rewriting eq. (2.97)

$$\tilde{\nabla}_b J_{aij} = \underset{\substack{\downarrow \\ ① \\ \downarrow \\ ②+③ \\ \downarrow \\ ④+⑤+⑥}}{D_b J_{aij}} \quad - \gamma_{ba}{}^{c} J_{cij} \quad - w_{bi}{}^{k} J_{akj} \quad - w_{bj}{}^{k} J_{aik}.$$

Substituting eqs. (2.101), (2.106), (2.108) and ⑥ $= g\left(N_j, R(E_b, N_i)E_a\right)$ into above equation we get

$$\begin{aligned} \tilde{\nabla}_b J_{aij} &= -\left[D_i K_{baj} + K_{ba}{}^{k}\gamma_{ikj} - K_{b}{}^{c}{}_{j} S_{aci} - K_{caj} S_{b}{}^{c}{}_{i}\right] \\ &- J_{akj} J_{bi}{}^{k} - K_{caj} K_{b}{}^{c}{}_{i} + g\left(N_j, R(E_b, N_i)E_a\right) \\ &+ \gamma_{ba}{}^{c} J_{cij} + w_{bj}{}^{k} J_{aik} + J_{akj} w_{bi}{}^{k} - \gamma_{ba}{}^{c} J_{cij} - w_{bi}{}^{k} J_{akj} - w_{bj}{}^{k} J_{aik}. \end{aligned} \tag{2.109}$$

Now due to relations (2.86) and (2.87) we have

$$\tilde{\nabla}_i K_{abj} = \underbrace{\nabla_i K_{abj}}_{D_i K_{abj} - \gamma_{ijk} K_{ab}{}^{k}} \quad - S_{aci} K_{b}{}^{c}{}_{j} - S_{bci} K_{a}{}^{c}{}_{j}. \tag{2.110}$$

Moreover K_{baj} is symmetric with respect to the first two indices and γ_{ikj} is antisymmetric with respect to the last two indices. Therefore the first line of eq. (2.109) can simply be written as $-\tilde{\nabla}_i K_{abj}$. Also the terms that appear on the last line of eq. (2.109) cancel each other. Then finally we obtain the Raychaudhuri equation of the worldsheet as

$$\left(\tilde{\nabla}_b J_{aij}\right) = -\left(\tilde{\nabla}_i K_{abj}\right) - J_{bik} J_{a}{}^{k}{}_{j} - K_{bci} K_{a}{}^{c}{}_{j} + g(R(E_b, N_i)E_a, N_j), \tag{2.111}$$

with $g(R(E_b, N_i)E_a, N_j) = R_{\alpha\beta\mu\nu} E^{\mu}{}_{b} N^{\nu}{}_{i} E^{\beta}{}_{a} N^{\alpha}{}_{j}$. After we contract the Raychaudhuri equation with the orthogonal basis metrics η^{ab} and δ^{ij} we get

$$\left(\tilde{\nabla}_b J_{aij}\right)\eta^{ab}\delta^{ij} = -\left(\tilde{\nabla}_i K_{abj}\right)\eta^{ab}\delta^{ij} - J_{bik} J_{a}{}^{k}{}_{j}\eta^{ab}\delta^{ij} - K_{bci} K_{a}{}^{c}{}_{j}\eta^{ab}\delta^{ij} + g(R(E_b, N_i)E_a, N_j)\eta^{ab}\delta^{ij}. \tag{2.112}$$

This is the central equation which we will refer to many times throughout the thesis.

We finalize the Preliminaries chapter here. Mathematical constructions and concepts that are introduced in this chapter will be used to investigate quasilocal systems that are in thermodynamic equilibrium and in nonequilibrium in the following two chapters.

3 Quasilocal equilibrium thermodynamics

A thermodynamic description of general relativity has been a long-sought goal [55, 56] which intensified with the advent of black hole mechanics [57, 58, 59]. Most studies in the literature focus on *equilibrium* thermodynamics[1] of *horizons*, without stating the conditions that bring them into equilibrium. In fact, in gravitational physics there are no well-defined conditions for defining equilibrium in terms of the behaviour of a *system* itself. In this chapter we will take steps to remedy this by defining a quasilocal thermodynamic equilibrium condition using a purely geometric approach. We will focus on the extrinsic geometry of the closed spacelike 2–surface that appears both in various quasilocal energy definitions [25, 26, 27] and the generalized Raychaudhuri equation of Capovilla and Guven [32] that we have reviewed in Section 2.1 and in Section 2.2 respectively.

In general, if one wants to investigate the energy exchange mechanisms of a gravitational system from the thermodynamic viewpoint, the system should have a finite spatial size. Energy definitions which refer to the spatial asymptotic behaviour are not good candidates for general thermodynamic equations. Thus quasilocal energy definitions, which refer to a Hamiltonian on the 2-dimensional spacelike boundary [24, 25], are very important for general relativistic thermodynamics. Here we will link such definitions to a generalized notion of the work done in the deviation of worldsheet congruences, to define quasilocal thermodynamic potentials in a natural way. This will help us to define the quasilocal thermodynamic equilibrium. We also provide a quasilocal first law by considering a *worldsheet total variation*, in which a quasilocal temperature can be understood as a *worldsheet–constant*.

Although early investigations dealt with equilibrium thermodynamics of black hole

[1] See [60, 61, 62, 63] for some exceptions.

event horizons, in the last few decades more general trapping, apparent and dynamical horizons of generic spacetimes have been introduced [16, 19, 22, 64, 49]. Hayward's construction of equilibrium thermodynamics on trapping horizons [16] highlights the significance of generalized apparent horizons. While the original definition of apparent horizons applies to black holes which require asymptotically flat spatial hypersurfaces [15], Hayward's generalized apparent horizon which was first constructed for black holes, has also been applied in more general cases [65, 49, 66]. These include cosmological applications, where the generalized apparent horizon is not necessarily spacelike but can be timelike or null depending on the equation of state of the cosmic fluid [49].

In this chapter we will consider a spherically symmetric gravitational system of arbitrary size which is not in equilibrium with its surroundings. As one of our results we will show that when a particular equilibrium condition is applied to such a system then the 2–surface enclosing the system is located at the generalized apparent horizon of [16]. This result makes no direct reference to the surface gravity, which is conventionally used to define the temperature of the horizon.

This chapter is constructed as follows. In Section 3.1 we remind the reader about the generalized Raychaudhuri equation of Capovilla and Guven [32] and our motivation to use it as a quasilocal thermodynamic relation. Following this, a quasilocal thermodynamic equilibrium condition and the corresponding thermodynamic potentials are introduced. In Section 3.2 these results are applied to the Schwarzschild, Friedmann-Lemaître-Robertson-Walker and Lemaître-Tolman spacetimes. In Section 3.3 we highlight the difference between local thermodynamics of matter fields on curved background and quasilocal gravitational thermodynamics, as a precursor to suggesting a potential application of our approach to a quasilocal virial relation in Section 3.4.

3.1 Raychaudhuri equation and gravitational thermodynamics

Recall that in Section 2.2 we introduced the geometric construction developed by Capovilla and Guven [32] in order to generalize the Raychaudhuri equation. This equation, (2.112), gives the focusing of an arbitrary dimensional timelike worldsheet that is embedded in an arbitrary dimensional spacetime. Also in Section 2.1 we observed that the extrinsic curvature of a closed spacelike 2-surface when it is perturbed along *both* a timelike and a spacelike direction gives information about the quasilocal matter plus gravitational energy of the system that it encloses.

When we apply the CG formalism to a 2-dimensional timelike worldsheet, $\mathbb{T}$, embedded in a 4-dimensional spacetime we observe that the Raychaudhuri equation of Capovilla and Guven includes those terms related to the extrinsic curvature of a closed spacelike 2-surface. This surface, $\mathbb{S}$, is in fact the normal frame of the worldsheet $\mathbb{T}$ and its extrinsic geometry is closely related to how much the worldsheet focuses. This gives us enough motivation to start investigating the Raychaudhuri equation in the $2+2$ formalism on the basis of quasilocal matter plus gravitational energy.

We presented a derivation of the generalized Raychaudhuri equation of Capovilla and Guven, (2.112), in Section 2.2 and ended up with the following contracted Raychaudhuri equation of [32], which for convenience we rewrite as

$$-\tilde{\nabla}_{\mathbb{T}}\mathcal{J} = \tilde{\nabla}_{\mathbb{S}}\mathcal{K} + \mathcal{J}^2 + \mathcal{K}^2 - \mathcal{R}_{\mathcal{W}}, \tag{3.1}$$

where we denote $\tilde{\nabla}_{\mathbb{T}}\mathcal{J} := \eta^{ab}\delta^{ij}\tilde{\nabla}_b J_{aij}$, $\tilde{\nabla}_{\mathbb{S}}\mathcal{K} := \eta^{ab}\delta^{ij}\tilde{\nabla}_i K_{abj}$, $\mathcal{J}^2 := J_{bik}J_{alj}\eta^{ab}\delta^{ij}\delta^{lk}$, $\mathcal{K}^2 := K_{bci}K_{adj}\eta^{ab}\eta^{cd}\delta^{ij}$, $\mathcal{R}_{\mathcal{W}} := g(R(E_b,N_i)E_a,N_j)\eta^{ab}\delta^{ij}$. This form, eq. (3.1), will be the central equation of our investigation.

Now let us consider a general spherically symmetric spacetime. For radially moving observers, the extrinsic curvature and the extrinsic twist of $\mathbb{T}$ both vanish as well as the extrinsic twist of $\mathbb{S}$. Then the first and the third terms on the r.h.s of eq. (3.1) vanish and the equation reduces to

$$-\tilde{\nabla}_{\mathbb{T}}\mathcal{J} = \mathcal{J}^2 - \mathcal{R}_{\mathcal{W}}, \tag{3.2}$$

We will now interpret eq. (3.2) as a thermodynamic relation for a quasilocally defined spherically symmetric system while presenting our motivation for doing so.

3.1.1 Quasilocal thermodynamic equilibrium conditions

At this point one needs to take care with the definitions of the thermodynamic variables under equilibrium and nonequilibrium conditions. In general, equilibrium can be seen as a specific state of a system that is ordinarily in nonequilibrium with its surroundings. Given specific equilibrium conditions, the equilibrium state acts as an *attractor* to bring the system into a preferably stable state [67]. Moreover, in classical thermodynamics, only in the equilibrium case are the existence of thermodynamic potentials guaranteed [68]. For such equilibrium states the thermodynamic potentials are Lyapunov functions[2], and they can be written as *linear* combinations of each other [70].

Let us recall the definitions of thermodynamic potentials in classical thermodynamics at the equilibrium state [71],

$$Helmholtz\ Free\ Energy : \mathcal{F} \doteq \mathcal{U} - \mathcal{T}\mathcal{S}, \tag{3.3}$$

$$Gibbs\ Free\ Energy : \mathcal{G} \doteq \mathcal{F} + \mathcal{W}, \tag{3.4}$$

$$Enthalpy : \mathcal{H} \doteq \mathcal{G} + \mathcal{T}\mathcal{S} \doteq \mathcal{U} + \mathcal{W}, \tag{3.5}$$

where $\mathcal{U}$ is the internal energy, and $\mathcal{W}$ represents the work terms which may include PV type and other types of work in general. The '$\doteq$' sign will be used for equations that hold *only* at quasilocal thermodynamic equilibrium from now on.

Note that Helmholtz free energy is the amount of reversible work done on a system in an isothermal process [72]. It is one of the thermodynamic variables that can be defined both in equilibrium and in nonequilibrium states [73]. Moreover, one way of defining the thermodynamic equilibrium is to set the Helmholtz free energy to its minimum value [70]. For this reason the Helmholtz free energy provides a physically natural means to interpret eq. (3.2) thermodynamically.

[2] Lyapunov functions are nonnegative functions that have at least one local maxima or minima at a point of interest. They are continuous functions with continuous first order derivatives and they vary monotonically with the evolution parameter [69].

3.1.1.1 Helmholtz free energy density

We will take the extrinsic curvature scalar of $\mathbb{S}$ as a measure of the matter plus gravitational Helmholtz free energy density and define[3]

$$f^a f_a := 2\mathcal{J}^2. \tag{3.6}$$

since

$$f := \sqrt{f^a f_a} = \sqrt{k^2 - l^2}. \tag{3.7}$$

is the object that appears in the quasilocal energy definitions of Section 2.1. Note that one can relate $J_{bik} J_{alj} \eta^{ab} \delta^{ij} \delta^{lk} := \mathcal{J}^2$ and $k^2 - l^2$ through $2\mathcal{J}^2 = k^2 - l^2$ on account of specific choices of double dyad vectors $\{E^{\mu}{}_a, N^{\mu}{}_i\}$ that exist naturally in spherically symmetric systems for radially moving observers. We will discuss more about this in Chapter 4 and provide a generic relation between $\mathcal{J}^2$ and $k^2 - l^2$ for arbitrary spacetimes. Then the Helmholtz free energy of the system is obtained once f is integrated on $\mathbb{S}$, i.e.,

$$\mathcal{F} = \frac{1}{16\pi} \oint_{\mathbb{S}} f \cdot d\mathbb{S}. \tag{3.8}$$

Since the equilibrium condition is defined in this case by the minimum of the Helmholtz free energy, other thermodynamic potentials should be written as linear combinations of each other once one sets $\mathcal{F} = \mathcal{F}_{min}$. Thus at equilibrium, the Gibbs free energy and the internal energy should read

$$\mathcal{G} \doteq \mathcal{F}_{min} + \mathcal{W}, \tag{3.9}$$

$$\mathcal{U} \doteq \mathcal{F}_{min} + \mathcal{T}\mathcal{S}. \tag{3.10}$$

where $\mathcal{W}$, $\mathcal{T}$ and $\mathcal{S}$ are to be defined. The Helmholtz free energy density defined by eq. (3.6) and eq. (3.7) is required to be a nonnegative real scalar, and the minimum value it can take is zero. This brings us to write the equations above with $\mathcal{F}_{min} = 0$ as

$$\mathcal{G} \doteq \mathcal{W}, \tag{3.11}$$

$$\mathcal{U} \doteq \mathcal{T}\mathcal{S}. \tag{3.12}$$

[3]The $\mathcal{J}^2/4$ term appears in the definition of the Hawking [74] and Liu-Yau [27] mass-energies, as the term $\mu\rho$ in the notation of these authors.

Then when $f = \sqrt{f^a f_a} = \sqrt{k^2 - l^2} \doteq 0$, eq. (3.2) becomes

$$-\tilde{\nabla}_{\mathbb{T}}\mathcal{J} = -\mathcal{R}_{\mathcal{W}}, \tag{3.13}$$

Recalling the fact that thermodynamic potentials are nonnegative real functions at equilibrium, we will force the terms in eq. (3.13) to take nonnegative values by taking the absolute value of each side before we start making further quasilocal thermodynamic interpretations. Thus,

$$\left|-\tilde{\nabla}_{\mathbb{T}}\mathcal{J}\right| = \left|-\mathcal{R}_{\mathcal{W}}\right|. \tag{3.14}$$

3.1.1.2 Work density

We will now give a thermodynamic interpretation to the quantity on the r.h.s. of eq. (3.14). In the 3+1 formalism, when one considers two observers on neighbouring timelike geodesics the deviation of the geodesics determines the relative accelerations of the observers. If we consider the spacelike separation 4-vector, $\vec{\xi}$, that connects the neighbouring geodesics, then the components of the relative tidal acceleration are given by [12]

$$\frac{d^2\xi^\mu}{d\tau^2} = R^\mu{}_{\nu\rho\sigma}u^\nu u^\rho \xi^\sigma, \tag{3.15}$$

where τ is the proper time. Thus for a spherically symmetric spacetime we define a *relative work density* term that mimics $W = \vec{F}\cdot\vec{x}$ by

$$\left(\frac{d^2\xi^\mu}{d\tau^2}\right)\xi_\mu = R^\gamma{}_{\nu\rho\sigma}u^\nu u^\rho \xi^\sigma \xi_\gamma. \tag{3.16}$$

This relative work density term can be interpreted as a measure of energy expended within the surface of a body to stretch or contract it under the influence of tidal forces, if we assume $\vec{\xi}$ lives on the screen, $\mathbb{S}$.

Our interpretation is similar to that of Schutz [75] who considered the limits of validity of the geodesic deviation equation and calculated the second order contributions. He also acknowledged the fact that connecting two geodesics with a separation vector is essentially nonlocal. Thus the reason eq. (3.15) is valid only for nearly parallel, neighbouring geodesics is simply due to observers trying to measure a nonlocal quantity, locally. Consequently eq. (3.16) has a more fundamental quasilocal interpretation and the $\left|-g(R(E_b, n^j)E^b, n_j)\right| = \left|-\mathcal{R}_{\mathcal{W}}\right|$ term on the r.h.s. of eq. (3.14) might be taken as a measure of *work density* attributed to $\mathbb{S}$.

To understand this intuitively, consider the analogy of a soap bubble. The work done per area to create the surface of a bubble is [71]

$$\mathcal{W}_{class} = \oint \gamma \cdot dA, \tag{3.17}$$

where γ is the surface tension and dA is a surface area element of the bubble. According to classical theory, surface tension arises due to the unbalanced intermolecular forces in the bubble. Likewise, according to the analogy formed here, $\left|-\mathcal{R}_{\mathcal{W}}\right|$ is a measure of energy density due to the relative tidal forces that observers experience when they move along radial worldlines. This is of course applicable for observers who share the same screen $\mathbb{S}$. Therefore at quasilocal thermodynamic equilibrium, we can define a general work density, namely a type of surface tension, according to

$$w := \sqrt{w^a w_a} := \sqrt{2\left|-\mathcal{R}_{\mathcal{W}}\right|}, \tag{3.18}$$

so that the amount of corresponding work is given by

$$\mathcal{W} \doteq \frac{1}{16\pi} \oint_{\mathbb{S}} w \cdot d\mathbb{S}. \tag{3.19}$$

3.1.1.3 Gibbs free energy density

In classical thermodynamics, when equilibrium is defined by the minimum of the Helmholtz free energy, the Gibbs free energy reads [71]

$$\mathcal{G}_{class} \doteq \oint \gamma \cdot dA, \tag{3.20}$$

for the thermodynamics of surfaces with constant pressure. Following the analogy with the surface of a soap bubble,

$$\mathcal{W}_{class} = \oint \gamma \cdot dA \Leftrightarrow \mathcal{W} \doteq \oint_{\mathbb{S}} w \cdot d\mathbb{S}, \tag{3.21}$$

so that

$$\mathcal{G}_{class} \doteq \oint \gamma \cdot dA \Leftrightarrow \mathcal{G} \doteq \mathcal{W} \tag{3.22}$$

should hold. This is consistent with eq. (3.11) which states that $\mathcal{G} \doteq \mathcal{W}$ since the minimum Helmholtz free energy is zero according to the equilibrium condition defined here. Thus, the l.h.s. of eq. (3.14) can be taken as a measure of the quasilocal Gibbs free energy density:

$$\mathscr{g} := \sqrt{\mathscr{g}^a \mathscr{g}_a} := \sqrt{2\left|-\tilde{\nabla}_{\mathbb{T}}\mathscr{J}\right|}, \tag{3.23}$$

with $\tilde{\nabla}_{\mathbb{T}}\mathscr{J} = \eta^{ab}\delta^{ij}\tilde{\nabla}_b J_{aij}$ from which the Gibbs energy can be obtained by

$$\mathcal{G} \doteq \frac{1}{16\pi}\oint_{\mathbb{S}} \mathscr{g} \cdot d\mathbb{S}. \tag{3.24}$$

Note that, in general, the Raychaudhuri equation becomes nonlinear if one wants to write it in terms of the energy densities defined here. However, recall that the existence of thermodynamic potentials is guaranteed only in the equilibrium case in which the potentials can be written linearly in terms of each other. In classical surface thermodynamics, the surface tension, pressure gradient across the surface and mean curvature of the surface can be related via the Young-Laplace equation [76]. Ideally, in order to reach the equilibrium, fluids tend to extremize their surfaces until they have zero mean curvature. This is when the the surface tension takes its critical value. In the formalism presented here, which is in line with our analogy, this happens when $f = \sqrt{f^a f_a} = \sqrt{k^2 - l^2} \doteq 0$, which defines the apparent horizon of a given spacetime.

Here we use a general apparent horizon [16], defined by the marginal surfaces on which at least one of the expansion scalars of the null congruences is zero, i.e., $\theta_{(l)}\theta_{(n)} = 0$, where l^a (n^a) is the outward (inward) pointing future-directed normal. Both the conditions $\{\theta_{(l)} > 0, \theta_{(n)} = 0\}$ and $\{\theta_{(l)} = 0, \theta_{(n)} < 0\}$ have previously been used to define apparent horizons [49, 65, 66]. Here $J_a{}^{ij} J^a{}_{ji}$ in eq. (3.6) gives a measure of $\theta_{(l)}\theta_{(n)}$. Thus when it is equated to zero, one can conclude that at least one of the expansion scalars of the incoming or outgoing null congruences converges without knowing which one actually does.

3.1.1.4 Internal energy density

On introducing the quasilocal energy definitions in Section 2.1, we stated that $E_{\mathrm{K}1}$ of Kijowski is a good candidate for the total matter plus gravitational energy content of a system. It is derived via a Hamilton-Jacobi formalism with Dirichlet boundary conditions. According to Kijowski those boundary conditions are associated with the true degrees of freedom of the quasilocally defined domain that gives the *true energy* [25]. When the equilibrium condition is imposed, the internal energy density in eq. (2.47) can be written as

$$u \doteq k_0. \tag{3.25}$$

Thus the quasilocal internal energy at equilibrium becomes

$$\mathcal{U} \doteq \frac{1}{16\pi} \oint_{\mathbb{S}} k_0 \cdot d\mathbb{S}. \tag{3.26}$$

which should satisfy the equilibrium condition (3.12) without any PV type term. This requires that we define a quasilocal entropy and temperature at equilibrium.

Since the 2-surface $\mathbb{S}$ located at the generalized apparent horizon enters naturally, we can follow the traditional approach of Bardeen [59] and define the quasilocal equilibrium entropy as

$$\mathcal{S} \doteq \frac{Area\,(\mathbb{S})}{\lambda}, \tag{3.27}$$

where λ is a constant which is usually taken to be 4 in gravitational equilibrium thermodynamics of horizons. By eq. (3.12), at equilibrium the quasilocal temperature of the system is then given by

$$\mathcal{T} \doteq \frac{\mathcal{U}}{\mathcal{S}}, \tag{3.28}$$

without any direct reference to surface gravity.

3.1.1.5 First law of thermodynamics

According to the formalism constructed here, the first law should be written as

$$\delta\mathcal{U} \doteq \delta(\mathcal{T}\,\mathcal{S}) \doteq \mathcal{T}\,\delta\mathcal{S}, \tag{3.29}$$

since we defined the quasilocal thermodynamic equilibrium via the minimization of the Helmholtz free energy which is applicable for isothermal processes. The problem with some of the gravitational thermodynamic constructions is that the total variation of the internal energy and entropy is performed in specific directions for which the first law does not have the dimensions of energy[4]. However, in our framework the quasilocal behaviour of the system sets the degrees of freedom with respect to which

[4]For example see [49].

the total variation can be defined. These are the degrees of freedom that reside on the instantaneously defined timelike surface $\mathbb{T}$. Hence, we will set the total variation to be the one on the worldsheet and write

$$\delta\mathcal{U} := \frac{1}{2}\sqrt{\tilde{\nabla}_a\mathcal{U}\tilde{\nabla}^a\mathcal{U}}. \tag{3.30}$$

Thus the first law should read

$$\frac{1}{2}\sqrt{\tilde{\nabla}_a\mathcal{U}\tilde{\nabla}^a\mathcal{U}} \doteq \mathcal{T}\left(\frac{1}{2}\sqrt{\tilde{\nabla}_a\mathcal{S}\,\tilde{\nabla}^a\mathcal{S}}\right), \tag{3.31}$$

with

$$\delta\mathcal{T} := \frac{1}{2}\sqrt{\tilde{\nabla}_a\mathcal{T}\,\tilde{\nabla}^a\mathcal{T}} \doteq 0, \tag{3.32}$$

where $\{a,b\} = \{\hat{0},\hat{1}\}$. Thus the temperature will be a *worldsheet–constant* rather than a constant with respect to some coordinate time. For the examples that are presented in the next section, it is easy to check that eq. (3.32) is satisfied. Note that, in general,

$$\tilde{\nabla}_a \to \nabla_a \to D_a \to E^\mu{}_a D_\mu \to E^\mu{}_a\partial_\mu, \tag{3.33}$$

can be used for the scalar functions that appear in equations (3.31) and (3.32).

3.2 Examples

3.2.1 Schwarzschild geometry

Consider the Schwarzschild metric in standard coordinates

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{3.34}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. As stated previously, to define a consistent system observers should have fixed angular coordinates. As one example consider static radial observers with double diad

$$E^{\mu}_{\ \hat{0}} = \left(\frac{1}{\sqrt{\left(1 - \frac{2M}{r}\right)}}, 0, 0, 0 \right), \qquad E^{\mu}_{\ \hat{1}} = \left(0, \sqrt{\left(1 - \frac{2M}{r}\right)}, 0, 0 \right), \tag{3.35}$$

$$N^{\mu}_{\ \hat{2}} = \left(0, 0, \frac{1}{r}, 0 \right), \qquad N^{\mu}_{\ \hat{3}} = \left(0, 0, 0, \frac{1}{r \sin\theta} \right). \tag{3.36}$$

The choice of static observers is inconsequential for our results, which also apply to observers with an arbitrary instantaneous radial boost with respect to this frame. For such observers, eq. (3.6) implies

$$f = \sqrt{f_a f^a} = 2\sqrt{\left(\frac{r - 2M}{r^3}\right)}, \tag{3.37}$$

which can be substituted in the Raychaudhuri equation, (3.2), in the general nonequilibrium case to give a notion of nonequilibrium quasilocal energy exchange.

In order to set the quasilocal thermodynamic equilibrium condition, $\mathcal{F}$ should be minimized. It is easy to see that, this occurs when $r = 2M$ which coincides with the location of the black hole horizon. Now let us calculate the internal energy at equilibrium. Given the metric in (3.34) and the isometric embedding of $\mathbb{S}$ into Euclidean 3-space, one finds $k_0 = 2/r$. Then according to equations (3.12) and (3.26),

$$\mathcal{U} \doteq \mathcal{T}\mathcal{S} \doteq M, \tag{3.38}$$

with $\mathcal{T} \doteq \lambda/(8\pi r) \doteq \lambda/(16\pi M)$, and $\mathcal{S} \doteq Area(\mathbb{S})/\lambda \doteq (16\pi M^2)/\lambda$ where λ is a constant. For those who wish to relate this temperature to the Hawking temperature [77], there is a problem of factor of two, which has been encountered in similar context before [78, 79, 80, 81, 82]. For $\lambda = 4$ the temperature gives twice the Hawking temperature, i.e., $\mathcal{T} = 2T_H = 2(8\pi M)^{-1}$. The literature is divided into two camps when it comes to the value of the temperature of radiation for a particle that tunnels through the horizon. Usually, those who favour $\mathcal{T} = 2T_H$ also favour the idea of dividing the entropy by 2 in order to satisfy Hawking's original first law [80, 82]. However, according to Hawking's original first law [77], for a static black hole [83],

$$\frac{Energy}{2} = Temperature \times Entropy. \tag{3.39}$$

Thus, if $\lambda = 4$ then one should *not* divide the original entropy expression by 2 in order to get the correct internal energy on the l.h.s. of eq. (3.39).

Also it is easy to check eq. (3.32),

$$\delta\mathcal{T} = \frac{1}{2}\sqrt{\tilde{\nabla}_a\left(\frac{\lambda}{8\pi r}\right)\tilde{\nabla}^a\left(\frac{\lambda}{8\pi r}\right)} \doteq 0. \tag{3.40}$$

This states that the system can be assigned a single temperature value which is a worldsheet–constant. One also finds the corresponding work term, (3.19), and Gibbs free energy term, (3.24),

$$\mathcal{W} \doteq \frac{1}{16\pi}\sqrt{2\left|\frac{-4M}{r^3}\right|}\left(4\pi r^2\right), \qquad \mathcal{G} \doteq \frac{1}{16\pi}\sqrt{2\left|\frac{2(r-4M)}{r^3}\right|}\left(4\pi r^2\right) \tag{3.41}$$

which are equal at the quasilocal thermodynamic equilibrium, i.e.,

$$\mathcal{G} \doteq \mathcal{W} \doteq M. \tag{3.42}$$

3.2.2 Friedmann-Lemaître-Robertson-Walker (FLRW) geometry

Now consider the FLRW metric in comoving coordinates

$$ds^2 = -dt^2 + \left(\frac{a^2(t)}{1-\kappa r^2}\right)dr^2 + a^2(t)r^2 d\Omega^2, \tag{3.43}$$

where $\kappa = \{-1, 0, 1\}$ for open, flat and closed universes respectively and the comoving observer dyads are

$$E^{\mu}{}_{\hat{0}} = (1,0,0,0), \qquad E^{\mu}{}_{\hat{1}} = \left(0, \frac{\sqrt{1-\kappa r^2}}{a}, 0, 0\right), \tag{3.44}$$

$$N^{\mu}{}_{\hat{2}} = \left(0,0,\frac{1}{ar},0\right), \qquad N^{\mu}{}_{\hat{3}} = \left(0,0,0,\frac{1}{ar\sin\theta}\right). \tag{3.45}$$

Again note that the resultant thermodynamic potential densities do not change for observers with an arbitrary instantaneous radial boost with respect to the comoving observers. For such a set up, Helmholtz free energy density is

$$f = \sqrt{f_a f^a} = \sqrt{2\left(\frac{2-2\kappa r^2-2\dot{a}^2 r^2}{a^2 r^2}\right)}. \tag{3.46}$$

If we consider the equilibrium case where free energy takes its minimum value, one can find the equilibrium condition to be

$$r \doteq \frac{1}{\sqrt{\kappa + \dot{a}^2}}, \qquad or \qquad r_A \doteq (ar) \doteq \frac{1}{\sqrt{H^2 + \kappa/a^2}}, \tag{3.47}$$

where H is the Hubble parameter. This corresponds to the location of the apparent horizon of the FLRW geometry [64, 49].
The internal energy density is found by isometrically embedding $\mathbb{S}$ into an Euclidean 3-geometry and calculating its extrinsic curvature as $k_0 = 2/(ar)$. Thus according to equations (3.12) and (3.26),

$$\mathcal{U} \doteq \mathcal{T}\mathcal{S} \doteq \frac{1}{2}\frac{1}{\sqrt{H^2 + \kappa/a^2}}, \tag{3.48}$$

with $\mathcal{T} \doteq \lambda/(8\pi ar) \doteq \lambda\sqrt{H^2 + \kappa/a^2}/(8\pi)$, and $\mathcal{S} \doteq Area(\mathbb{S})/\lambda \doteq 4\pi/\left[\lambda\left(H^2 + \kappa/a^2\right)\right]$.

For $\lambda = 4$, this result matches the one of [64, 84, 85, 86], where the temperature attributed to the apparent horizon is found to be $T_A = 1/(2\pi r_A)$.

This temperature, assigned to the whole system, can be shown to be a worldsheet–constant by the variation

$$\delta\mathcal{T} = \frac{1}{2}\sqrt{\tilde{\nabla}_a\left(\frac{\lambda}{8\pi ar}\right)\tilde{\nabla}^a\left(\frac{\lambda}{8\pi ar}\right)} \doteq 0. \tag{3.49}$$

When this condition holds, the work term (3.19) and the Gibbs free energy (3.24) are also found to be

$$\mathcal{W} \doteq \frac{1}{16\pi}\sqrt{2\left|\frac{2\kappa + 2a\ddot{a} + 2\dot{a}^2}{a^2}\right|}\left(4\pi a^2 r^2\right), \qquad \mathcal{G} \doteq \frac{1}{16\pi}\sqrt{2\left|\frac{2\ddot{a}}{a} + \frac{2}{r^2 a^2}\right|}\left(4\pi a^2 r^2\right). \tag{3.50}$$

By (3.47),

$$\mathcal{G} \doteq \mathcal{W} \doteq \frac{1}{2}\sqrt{\left|\frac{\kappa + a\ddot{a} + \dot{a}^2}{a^2}\right|}\frac{1}{H^2 + \kappa/a^2} \tag{3.51}$$

so that when the Friedmann equations are inserted we obtain

$$\mathcal{G} \doteq \mathcal{W} \doteq \frac{1}{4}\sqrt{\left|\frac{1 - 3p/\rho}{\frac{4\pi\rho}{3}}\right|}, \tag{3.52}$$

where p is the pressure, ρ is the energy density of the perfect fluid and their ratio is $\omega = p/\rho$. Alternatively, we can compare the work required to create a quasilocal 2-

surface that encloses a system filled with either vacuum energy ($\omega = -1$), stiff matter ($\omega = 1$), dust ($\omega = 0$) or radiation ($\omega = 1/3$). For the same value of the perfect fluid energy density, at equilibrium, the results state

$$\mathcal{W}_{Vacuum} > \mathcal{W}_{Stiff} > \mathcal{W}_{Dust} > \mathcal{W}_{Radiation} = 0, \tag{3.53}$$

meaning that a system filled with stiff matter has a greater tendency to store the potential relative work than a system filled with dust or radiation. Note that the surface tension is independent of the spatial size of the system in a FLRW spacetime, consistent with the fact that the FLRW geometry models a homogeneous universe. To see the differences with an inhomogeneous universe one may consider the Lemaître-Tolman spacetime.

3.2.3 Lemaître-Tolman (LT) geometry

The LT metric can be written in the comoving coordinates as

$$ds^2 = -dt^2 + \left(\frac{R'^2(t,r)}{1+2K(r)}\right)dr^2 + R^2(t,r)d\Omega^2, \tag{3.54}$$

where $X' = \frac{\partial X}{\partial r}$ and $\dot{X} = \frac{\partial X}{\partial t}$ for an arbitrary function X. One can choose a comoving observer dual dyad

$$E^{\mu}{}_{\hat{0}} = (1,0,0,0), \qquad E^{\mu}{}_{\hat{1}} = \left(0, \frac{\sqrt{1+2K}}{R'}, 0, 0\right), \tag{3.55}$$

$$N^{\mu}{}_{\hat{2}} = \left(0,0,\frac{1}{R},0\right), \qquad N^{\mu}{}_{\hat{3}} = \left(0,0,0,\frac{1}{R\sin\theta}\right). \tag{3.56}$$

Then the Helmholtz free energy density for such an observer is given by

$$f = \sqrt{2\left(\frac{2+4K-2\dot{R}^2}{R^2}\right)}. \tag{3.57}$$

Minimizing $\mathcal{F}$ to obtain the condition for quasilocal thermodynamic equilibrium we find

$$\dot{R}^2 \doteq 1+2K, \tag{3.58}$$

which again gives the location of the generalized apparent horizon of the LT geometry

[87]. In the absence of the cosmological constant, the evolution equation for the LT spacetime may be written as

$$\dot{R}^2 = 2K + \frac{2M}{R}, \tag{3.59}$$

where $M = M(r)$ is an arbitrary function which is said to play the role of the active gravitational mass within a constant radius shell in LT solutions. Therefore, another way of defining the apparent horizon is $R(t,r) \doteq 2M(r)$.Then after computing the internal energy density as $k_0 = 2/R$ by equations (3.12) and (3.26), the internal energy becomes

$$\mathcal{U} \doteq \mathcal{T}\,\mathcal{S} \doteq M(r), \tag{3.60}$$

with $\mathcal{T} \doteq \lambda/(8\pi R) \doteq \lambda/(16\pi M)$, and $\mathcal{S} \doteq Area(\mathbb{S})/\lambda \doteq (16\pi M^2)/\lambda$. If we take $\lambda = 4$, then the temperature assigned to the system takes the same value as the temperature attributed to the apparent horizon in [88]. The work term (3.19) and the Gibbs free energy (3.24) are also found as:

$$\mathcal{W} \doteq \frac{1}{16\pi}\sqrt{2\left| -\frac{2K'}{RR'} + \frac{2\ddot{R}}{R} + \frac{2\dot{R}\dot{R}'}{RR'}\right|}\left(4\pi R^2\right), \tag{3.61}$$

$$\mathcal{G} \doteq \frac{1}{16\pi}\sqrt{2\left|\frac{2\ddot{R}}{R} - \frac{2\dot{R}^2}{R^2} + \frac{2(2K+1)}{R^2} - \frac{\left(2K' - 2\dot{R}\dot{R}'\right)}{RR'}\right|}\left(4\pi R^2\right). \tag{3.62}$$

Substituting eq. (3.59) and $R(t,r) \doteq 2M(r)$ into the equations above gives

$$\mathcal{G} \doteq \mathcal{W} \doteq M(r)/\sqrt{2}. \tag{3.63}$$

In contrast to the homogeneous cosmology, the surface tension in eq. (3.61) depends on the radial position of the quasilocal observers who define the inhomogeneous system.

One can check the thermodynamic potentials of the LT system reduce to those of the FLRW and Schwarzschild spacetimes in the appropriate limit. In particular, for [87]

$$R = a(t)r \qquad and \qquad M = \int 4\pi\rho R^2 R'\,dr \tag{3.64}$$

the relative work term (3.63) agrees with the dust case of eq. (3.52) for the FLRW geometry as expected. Likewise, if we take $R = r$ and use the spatial derivative of the evolution equation (3.59) to eliminate $K'(r)$ then at equilibrium eq. (3.61) agrees with

(3.41) for the Schwarzschild geometry. However, the general relative work (3.63) of LT differs from the Schwarzschild case (3.42), on account of the competing terms in eqs. (3.61) or (3.62). Recall that the Gibbs free energy is a measure of how much potential the system possesses to do work. In the static limit the second and third terms inside the square root in eq. (3.61) vanish. In that case, the system stores all of the gravitational energy due to the spatial term $-2K'/(RR')$ as potential work without being compelled to expend some of this energy as the system evolves in time.

3.3 Local versus quasilocal equilibrium

In Newtonian physics a system composed of particles which are in local thermodynamic equilibrium with each other is not always expected to be in global thermodynamic equilibrium. In the case of global equilibrium, one can assign single values of thermodynamic variables to the whole system. Likewise, in our case we have defined a quasilocal thermodynamic equilibrium condition so that one can assign a single temperature value to the whole system.

In general, there is no reason for a system in local (or global) equilibrium to be in hydrodynamic equilibrium as well. One requires additional conditions for them to coincide [89], so we should be careful to distinguish these concepts.

A typical example of the *local* thermodynamics of matter fields on a curved background is given by Gao [90] who generalizes early work of Sorkin, Wald and Zhang [91] to a generic perfect fluid. He investigates the connection between local thermodynamic equilibrium and hydrostatic equilibrium. Gao considers a collection of monatomic ideal gas particles with $p = p(T)$, $\rho = \rho(T)$ and $s = s(\rho, n)$ where s is the locally defined entropy density and n is the number density of the particles. By maximizing the local matter entropy, an equation for local hydrostatic equilibrium is obtained. This is not a unique way of defining the local thermodynamic equilibrium but this specific thermodynamic equilibrium condition also satisfies the local hydrostatic equilibrium. Also it is important to note that the fluid particles are not necessarily in a local thermal equilibrium here.

Now let us consider the analogous problem of the conditions under which *quasilocal* thermodynamic equilibrium and *quasilocal* hydrodynamic equilibrium coincide for a general system containing both matter and gravitational energy contributions. We will assume that locally defined condition given by Green, Schiffrin and Wald [89] for matter fields also holds in the quasilocal case. This requires the quasilocally defined entropy to have its extremum value with respect to a total variation defined by relation (3.31). In order to compute this one can consider a generic spherically symmetric spacetime metric

$$ds^2 = -A^2(r,t)dt^2 + B^2(r,t)dr^2 + R^2(r,t)d\Omega^2, \tag{3.65}$$

where $A(r,t)$ and $B(r,t)$ are arbitrary functions, $R(r,t)$ is the areal radius of $\mathbb{S}$, and our double dyad is

$$E^\mu{}_{\hat{0}} = \left(\tfrac{1}{A(r,t)},0,0,0\right), \qquad E^\mu{}_{\hat{1}} = \left(0,\tfrac{1}{B(r,t)},0,0\right), \tag{3.66}$$

$$N^\mu{}_{\hat{2}} = \left(0,0,\tfrac{1}{R(r,t)},0\right), \qquad N^\mu{}_{\hat{3}} = \left(0,0,0,\tfrac{1}{R(r,t)\sin\theta}\right). \tag{3.67}$$

Then, quasilocal Helmholtz free energy takes its minimum value when $A^2R'^2 \doteq B^2\dot{R}^2$, and the total variation of the quasilocally defined entropy, (3.27), at equilibrium is

$$\begin{aligned}
\delta\mathcal{S} &= \frac{1}{2}\sqrt{\tilde{\nabla}_a\mathcal{S}\tilde{\nabla}^a\mathcal{S}} \\
&= \frac{1}{2}\sqrt{E^\mu{}_a\partial_\mu\left(\frac{4\pi R^2}{\lambda}\right)\eta^{ab}E^\nu{}_b\partial_\nu\left(\frac{4\pi R^2}{\lambda}\right)} \doteq 0,
\end{aligned}$$

showing that the entropy is an extremum. This allows us to conclude that the quasilocal thermodynamic equilibrium and quasilocal hydrodynamic equilibrium should coincide. The interpretation of this result is crucial for the next section.

3.4 Quasilocal virial relation

Here we will sketch how the formalism above might be adapted to give a quasilocal virial condition which differs in character from previous attempts to define a virial theorem in general relativity [92, 93, 94, 95]. In previous studies only matter fields have been investigated, in which the central object is the energy-momentum tensor defined at each spacetime point. However, a full description of the virial theorem in general relativity should also include gravitational energy, which cannot be defined at a point due to the equivalence principle.

Ordinarily in classical mechanics the virial theorem is obtained by considering a system with motions confined to a finite region of space. If the potential energy is a homogeneous function of the coordinates, then the virial theorem gives a relation between the time averaged values of the total kinetic and potential energies [96]. For such a system one can define a virial function G by [97] $G(t) = \sum_i \vec{p}_i.\vec{r}_i$, where $\vec{r}_i$ are the coordinates and the $\vec{p}_i$ are the momenta of the particles in the system. If $G(t)$ is a bounded function, then the mean value of its time derivative is zero, i.e.,

$$\left\langle \frac{d}{dt}(\vec{p}_i.\vec{r}_i) \right\rangle = \left\langle \sum_i (\vec{p}_i.\vec{v}_i) \right\rangle + \left\langle \sum_i \left(\vec{r}_i.\dot{\vec{p}}_i\right) \right\rangle = 0. \tag{3.68}$$

Therefore, for a system under gravitational potential, the virial theorem reads

$$2\langle K.E\rangle = -\langle P.E\rangle, \tag{3.69}$$

where $\langle K.E\rangle$ and $\langle P.E\rangle$ are the time averaged values of the total kinetic and potential energies.

When it comes to including relativistic effects, however, this result changes. For the ultrarelativistic limit $(v \to c)$, the virial theorem takes the form [98]

$$\langle K.E\rangle = -\langle P.E\rangle. \tag{3.70}$$

In astrophysics, the virial theorem is used when the thermal and the gravitational forces acting on an isolated system balance each other so that the system neither expands nor contracts. This state of the system is defined by its hydrostatic equilibrium [99]. In general the system is assumed to be composed of ideal gas particles which are in local thermal equilibrium with each other, which guarantees stability [89]. The value of the internal energy of the system, E^{in}, is then equal to the ensemble average of the kinetic energies of the particles creating the system [100], i.e.,

$$E^{\text{in}} = \overline{K.E}, \tag{3.71}$$

where the overbar denotes the ensemble average at a given time. Note that the equipartition theorem is an application of the virial theorem. If thermal equilibrium coincides with hydrostatic equilibrium, then the temporal average of the kinetic energy of the total system becomes equal to the ensemble average of the kinetic energies of the particles at a given time [101], i.e., $\langle K.E\rangle|_{\{t_1\to t_2\}} \equiv \overline{K.E}|_{\{t\}}$. Consequently, for this case, one can rewrite (3.69) and (3.70) as

$$2E^{\text{in}} = -\langle P.E\rangle \qquad (nonrelativistic) \tag{3.72}$$

and [98]

$$E^{\text{in}} = -\langle P.E\rangle \qquad (ultrarelativistic) \tag{3.73}$$

In our case, quasilocal thermodynamic equilibrium is set by the minimum of the Helmholtz free energy which holds for systems with worldsheet–constant temperature and volume. This occurs when the mean extrinsic curvature of $\mathbb{S}$ is zero. Thus if one perturbs $\mathbb{S}$ along $\mathbb{T}$, it *neither expands nor contracts*. Furthermore, the quasilocally defined entropy of the system then takes its extremum value. We can then expect the quasilocal thermodynamic and hydrodynamic equilibria to coincide.

By analogy to the case of matter fields only, a virial theorem might hold also in our quasilocal formalism. In particular, we will suggest a natural bound,

$$E_{matter+grav} \doteq E_{\text{KLY}} - E_B^{min}. \tag{3.74}$$

from which a quasilocal virial relation follows. Here $E_{matter+grav}$ represents the combined energy of the matter and gravitational fields, while E_B^{min} is the minimum binding energy of the system. Furthermore, the interpretation of E_{KLY} as an invariant proper mass-energy for generic spacetimes is agreed upon by many authors [26, 27, 45]. We demonstrate that (3.74) agrees with known results in particular limits, but will not attempt a formal proof of this bound or consequently of a virial theorem, which would require a detailed definition of $E_{matter+grav}$ in terms of suitable ensemble averages.

We first recall that Bizon, Malec and Ó Murchadha [102] introduced a mass bound for a collection of spherical shells under collapse, given by

$$M \leq M_p - E_B, \tag{3.75}$$

where M is the total energy of the shells, M_p is the total proper mass and E_B is the binding energy. The equality holds when binding energy is minimum, which turns out to be the Newtonian limit. Later, Yu and Caldwell [103] included this argument in their calculation of the binding energy of a Schwarzschild black hole and showed that

$$M = M_p - E_B^{min}, \tag{3.76}$$

and $M_p = E_{\mathrm{BY}}$ in the Schwarzschild geometry for static observers at any r. Since $E_{\mathrm{BY}} = E_{\mathrm{KLY}}$ for static observers in the Schwarzschild geometry for observers who are instantaneously at rest, (3.76) is seen to coincide with (3.74) in the Schwarzschild geometry.

Now let us specialize to observers at quasilocal hydrostatic equilibrium in the Schwarzschild geometry, at $r = 2M$, where the Kijowski-Liu-Yau energy gives

$$E_{\mathrm{KLY}} = \frac{-1}{8\pi}\int_{\mathbb{S}} d\mathbb{S}\left(\frac{2\sqrt{1-\frac{2M}{r}}}{r} - \frac{2}{r}\right) \doteq 2M. \tag{3.77}$$

Since the internal energy, $E_{\mathrm{K}}^{\mathrm{in}} = E_{\mathrm{K}1}$ (2.47), at equilibrium corresponds to the usable matter plus gravitational energy, the l.h.s. of (3.74) should read

$$\begin{aligned} E_{matter+grav} &= E_{\mathrm{K}}^{\mathrm{in}} \\ &= \frac{-1}{16\pi}\int_{\mathbb{S}} d\mathbb{S}\left(\frac{\frac{4\left(1-\frac{2M}{r}\right)}{r^2} - \frac{4}{r^2}}{\frac{2}{r}}\right) \doteq M. \end{aligned}$$

Hence $E_B^{min} \doteq M$. The minimum binding energy of such a system can also be found by calculating the work done in bringing the spherical mass shells from infinity to $r = 2M$ by observers who are instantaneously at rest in the Newtonian limit [103]. Thus at hydrostatic equilibrium,

$$E_{matter+grav} \doteq E_B^{min}, \tag{3.78}$$

which is analogous to the virial relation (3.73) in the ultrarelativistic limit, since binding energy is negative of the potential energy, and only the so-called reference term survives in $E_{\mathrm{K}}^{\mathrm{in}}$ at quasilocal thermodynamic equilibrium.

Now we will generalize this result by *assuming* that (3.74) holds for *generic* spherically symmetric spacetimes at quasilocal hydrodynamic equilibrium. At quasilocal hydrodynamic equilibrium $\sqrt{2\mathcal{J}^2} = \sqrt{k^2 - l^2} \doteq 0$, by (2.47) and (2.56),

$$E_{matter+grav} \doteq E_{\mathrm{K}}^{\mathrm{in}} \doteq \frac{1}{16\pi}\oint_{\mathbb{S}} k_0\, d\mathbb{S}. \tag{3.79}$$

$$E_{\mathrm{KLY}} \doteq 2\left(\frac{1}{16\pi}\oint_{\mathbb{S}} k_0\, d\mathbb{S}\right), \tag{3.80}$$

Hence from eq. (3.74) we find

$$E_{\mathrm{KLY}} - E_{matter+grav} \doteq E_B^{min} \doteq \frac{1}{16\pi} \oint_{\mathbb{S}} k_0 \, d\mathbb{S}, \tag{3.81}$$

where $k_0 = 2/R(r,t)$ is the extrinsic curvature of $\mathbb{S}$ when embedded in Euclidean 3-space and $R(r,t)$ is the areal radius of $\mathbb{S}$. Therefore, by (3.79) and (3.81) one obtains (3.78) as a virial relation for any spherically symmetric distribution at quasilocal hydrodynamic equilibrium whose matter and gravitational contributions to the total content cannot be decoupled.

A key inference is that the proper mass–energy, usable matter–energy and the binding energy of a system make most sense when referred to measurements made by the same set of quasilocal observers. Therefore the idea of comparing certain energy definitions for systems with different sizes in a given spacetime can lead to paradoxes. For example, Frauendiener and Szabados argued that [104] *"...if the quasi-local mass* (E_{KLY}) *should really tend to the ADM mass as a strictly decreasing set function near spatial infinity, then the Schwarzschild example shows that the quasi-local mass at the event horizon cannot be expected to be the irreducible mass."* This is true simply because a system with a spatial size coinciding with the event horizon has different binding energy requirements than the one whose spatial size tends to infinity. The latter one has zero binding energy, because no work has to be done by the system defined by quasilocal observers located already at infinity. The authors continue with the statement *"...there would have to be a closed 2-surface between the horizon and the spatial infinity on which the quasi-local mass would take its maximal value. However, it does not seem why such a (geometrically, and hence, physically) distinguished 2-surface should exist."* Here we note that such a closed spatial 2-surface does exist with a location matching the apparent horizon. It encloses a system whose quasilocal thermodynamic equilibrium coincides with quasilocal hydrodynamic equilibrium. This might also serve as a physical interpretation for a generalized apparent horizon for the case when its location matches the one of a marginally outer trapped surface. The outgoing null rays of a system enclosed by such a trapped surface do not tend to leave the system because systems in quasilocal thermal equilibrium simply do not radiate.

3.5 Discussion

When the equilibrium thermodynamics *of horizons* was first introduced in the 1970s [57, 58, 59], the quasilocal energy definitions that we have today were unknown. It is now known that the physically relevant boundary Hamiltonian of general relativity lies on a closed 2-dimensional spacelike surface, $\mathbb{S}$, of a spacetime domain [24, 25], which we call the *screen*. In this chapter we have focused on a spherically symmetric *system* enclosed by a screen, $\mathbb{S}$, as the central object of gravitational thermodynamics rather than horizons.

Isolated systems are natural objects in classical thermodynamics. In general relativity, however, no system can be totally decoupled from the rest of the universe due to the nonlinear nature of the gravitational interaction. The systems we consider in this chapter have arbitrary size and are generally in nonequilibrium with their surroundings. Only after quasilocal thermodynamic equilibrium conditions are introduced does it follow that the screen is located at the apparent horizon of [16], where the standard equilibrium thermodynamic laws apply.

We believe that this approach may ultimately prove useful in general relativity, since the issues associated with quasilocal gravitational energy on one hand, or with gravitational entropy on the other, are generally studied in isolation. In fact, the problem of gravitational entropy is so complex that often researchers simply seek definitions in terms of geometric quantities which are nondecreasing with time [105, 106, 107], giving rise to a "second law", without directly investigating whether the entropies so defined obey any of the other properties one might reasonably demand of a genuine thermodynamic potential. The fact that the second law of classical thermodynamics can be viewed as a consequence of entropy not being rigorously defined in nonequilibrium [67] is usually overlooked.

On account of the equivalence principle, statistical macroscopic properties of the gravitational field are necessarily nonlocal. To interpret quasilocal gravitational energy in terms of thermodynamic laws it is necessary to have a measure of the "work done" by the tidal forces on the screen associated with the quasilocal observers. For this reason, we have adapted the generalized Raychaudhuri equation of an arbitrary dimensional worldsheet embedded in an arbitrary dimensional spacetime [32], to the special case of a 2-dimensional timelike surface, $\mathbb{T}$, (orthogonal to $\mathbb{S}$ at every point), which we embed directly into 4-dimensional spacetime.

The mean extrinsic curvature of $\mathbb{S}$, that appears in the quasilocal energy definitions [25, 26, 27], gives the expansion of $\mathbb{S}$ when it is perturbed along $\mathbb{T}$. Degrees of freedom residing on $\mathbb{T}$ are, therefore, understood to be those which describe the changes of thermodynamic potentials, while the degrees of freedom residing on the screen, $\mathbb{S}$, are required to consistently define a system. Hence, in order to write the first law, the total variations of the thermodynamic variables are taken along the 2-surface $\mathbb{T}$ rather than variations along the integral curve of a single vector.

It is known that quasilocal energy definitions which involve the mean extrinsic curvature of $\mathbb{S}$ are invariant under radial boosts. In our formalism this boost invariance holds for all thermodynamic potentials that appear in the generalized Raychaudhuri equation (3.1), once the definitions (3.6), (3.18) and (3.23) are made. This is possible on account of radially moving observers of a spherically symmetric spacetime.

Our spherically symmetric formalism might be easily extended to situations which are approximately spherically symmetric, in a perturbative scheme. Furthermore, we believe that similar reasoning to ours could also apply to spacetimes with other symmetries, such as axial symmetry. In that case one should be able to introduce additional quantities, which account for rotational energy, for example. In such a case the first and third terms on the r.h.s. of the generalized Raychaudhuri equation (3.1) are nonzero, making a thermodynamic interpretation considerably more complicated. We will discuss these issues in the next chapter.

As an application of our formalism, we have sketched a natural bound involving the quasilocal gravitational energy plus matter fields, which might suggest a virial relation. To rigorously prove a virial theorem requires that we have a proper understanding of the degenerate states of matter and gravitational fields contained within the screen, $\mathbb{S}$, which are consistent with the same worldsheet–constants on $\mathbb{S}$. Such an understanding of course requires going far beyond this thesis, as it effectively means probing fundamental questions related to the holographic interpretation which are important to both statistical and quantum gravity.

Other questions for future work relate to the question of nonequilibrium quasilocal gravitational thermodynamics for systems that are close to equilibrium. The Helmholz free energy, (3.8), is defined for all states of the system, whereas the other thermodynamic potentials have only been defined at quasilocal thermodynamic equilibrium. This is simply because equilibrium was defined by minimization of $\mathcal{F}$, which may not be the only way to define a useful quasilocal thermodynamic equilibrium condition. Other types of equilibrium conditions could be applied to the generalized Raychaudhuri equation.

For the case of thermodynamic nonequilibrium, the existence of thermodynamic variables is not guaranteed and their consistent definition becomes murky even in classical thermodynamics. Losing the linearity condition among the thermodynamic variables makes their interpretations much more difficult. However, the Raychaudhuri equation of the worldsheet (3.1) should still quantify the energy fluxes into and out of the system. That is what we will focus on in the next chapter.

Recently, Freidel [62] presented an approach to study nonequilibrium thermodynamics by using geometrical objects in a 2+2 formalism. Our language is similar to his in terms of the quasilocal nature of the thermodynamic potential densities introduced on a screen $\mathbb{S}$. However, our formalism differs fundamentally in character by the existence of a worldsheet $\mathbb{T}$ on which both of the degrees of freedom are treated equally in terms of their roles in evolving the potentials. Whether or not investigation of this difference provides a passage from quasilocal equilibrium to nonequilibrium thermodynamics is a point of interest.

In the next chapter, we will focus on systems that are not in quasilocal thermodynamic equilibrium. Moreover, we will relax the the condition of spherical symmetry. Our investigation will mainly involve the quasilocal energy exchange of the system. The thermodynamic interpretation, which does not exist for systems far from equilibrium even in classical thermodynamics, will be abandoned.

4 Quasilocal energy exchange and the null cone

In general relativity, there is no unique definition of matter plus gravitational energy exchange definition for a system. For the case of pure gravity, for example, gravitational radiation and the energy loss associated with it, can be identified unambiguously only at null infinity, $\Im^+$, of an isolated body [14]. Essentially it is assumed that observers are sufficiently far away from the body in question so that the asymptotic metric is flat and the perturbations around it correspond to the gravitational radiation. Also it is assumed that the spacetime admits the peeling property, i.e., the Weyl scalars behave asymptotically and outgoing null hypersurfaces are assumed to intersect $\Im^+$ through closed spacelike 2-surfaces whose departure from the unit sphere is small [108]. It is known that the wave extraction and the interpretation of the physically meaningful quantities are often challenging for numerical relativity simulations based on those asymptotic regions.

On the other hand, for astrophysical and larger scale investigations, we would like to know how systems behave in the strong field regime. We would like to understand the behaviour of binary black hole or neutron star mergers and how those objects affect their close environment. Considering the fact that gravitational energy cannot be localised due to the equivalence principle, there have been a considerable number of attempts to understand the energy exchange mechanisms of arbitrary gravitating systems quasilocally, on top of the earlier global investigations [109, 110, 111]. However, not all of the quasilocal energy investigations are constructed on, or translated into, the formalism that the numerical relativity community uses. In the present chapter, we aim to present a method with which one can investigate the quasilocal energy exchange of a system. This involves the observables of timelike congruences, however, we present the corresponding null cone observables as well once we perform a transformation between the two formalisms.

In Chapter 2 we presented Capovilla and Guven's generalized Raychaudhuri equation which gives the focusing of an arbitrary dimensional timelike worldsheet that is embedded in an arbitrary dimensional spacetime. Previously, in Chapter 3, we applied their formalism to a 2-dimensional timelike worldsheet, $\mathbb{T}$, embedded in a 4-dimensional spherically symmetric spacetime. This allowed us to define quasilocal thermodynamic equilibrium conditions and the corresponding quasilocal thermodynamic potentials in a natural way.

In the present chapter, we will consider more generic systems, which are not in equilibrium with their surroundings. Also the systems we consider here are not necessarily spherically symmetric. Our main aim is to present a method for the calculation of the energy-like quantities of these systems which can be exchanged quasilocally. While doing so, we will switch from Capovilla and Guven's notation to the notation of Newman-Penrose (NP) formalism [34]. Firstly, this will ease our calculations. Secondly, the transformation of the original formalism of CG to NP poses basic questions about the null tetrad gauge invariance of numerical relativity in terms of quasilocal concerns. Namely, if one wants to investigate a system quasilocally one needs to define it consistently throughout its evolution by keeping the boost invariance of the quasilocal observers. This fixes a gauge for the complex null tetrad constructed through their local double dyad in our 2+2 approach.

The construction of this chapter is as follows. In Section 4.1, we survey some of the local, global and quasilocal approaches in literature to investigate matter plus gravitational mass-energy exchange. We will show just how broad the literature is in terms of energy exchange investigations. In Section 4.2 we start to question how to best define a quasilocal system properly and introduce our choice of system definition. In Section 4.3 we present the contracted Raychaudhuri equation in the NP formalism and demonstrate how our gauge conditions affect it. Later, in Section 4.4, we give physical interpretations to the variables of the contracted Raychaudhuri equation in terms of the quasilocal charge densities. We define the associated quasilocal charges and end up with a work-energy relation. According to our interpretation, the contracted Raychaudhuri equation of the worldsheet of the quasilocal observers gives information about how much rotational and nonrotational quasilocal energy the system possesses, in addition to the work that should be done by the tidal fields to create such a system. In Section 4.5 we present applications of our method to a radiating Vaidya spacetime, C-metric and interior of a Lanczos-van Stockum dust source. We present the delicate issues related to our construction in Section 4.6 and give a summary and a discussion in Section 4.7. Our derivations, together with the relevant equations of the NP formalism , are presented in Appendices A, B and C.

Note that since we use $(-,+,+,+)$ signature for our spacetime metric in this thesis, one has to be careful about the definitions of the spin coefficients and curvature scalars when comparing them to Newman and Penrose's original construction in [34]. However, that is not a complication for our contracted Raychaudhuri equation as it is independent of the metric signature.

4.1 Mass-energy exchange: local, global and quasilocal

4.1.1 Local approaches

For local investigations of the gravitational energy flux, the Weyl tensor plays the central role. Newman and Penrose introduce five complex Weyl curvature scalars which incorporate all of the information of the Weyl tensor by [34]

$$\psi_0 = C_{\mu\nu\alpha\beta} l^\mu m^\nu l^\alpha m^\beta, \tag{4.1}$$

$$\psi_1 = C_{\mu\nu\alpha\beta} l^\mu n^\nu l^\alpha m^\beta, \tag{4.2}$$

$$\psi_2 = C_{\mu\nu\alpha\beta} l^\mu m^\nu \overline{m}^\alpha n^\beta, \tag{4.3}$$

$$\psi_3 = C_{\mu\nu\alpha\beta} l^\mu n^\nu \overline{m}^\alpha n^\beta, \tag{4.4}$$

$$\psi_4 = C_{\mu\nu\alpha\beta} n^\mu \overline{m}^\nu n^\alpha \overline{m}^\beta, \tag{4.5}$$

where $C_{\mu\nu\alpha\beta}$ is the Weyl tensor of the spacetime, $\{l_\mu, n_\mu, m_\mu, \overline{m}_\mu\}$ is the NP complex null tetrad and the only surviving inner products of the null vectors with each other are $\langle \mathbf{l}, \mathbf{n} \rangle = -1$ and $\langle \mathbf{m}, \overline{\mathbf{m}} \rangle = 1$.

The dynamics of timelike observers, who live in different Petrov-type spacetimes, was investigated by Szekeres previously [112]. In this method, one can assign physical meanings to the Weyl scalars. However, we note that this is only possible once we adapt our NP tetrad to the principal null direction(s) of the spacetime in question.

Once we relax this condition, Weyl curvature scalars cannot be interpreted as the way it was done in Szekeres' work.

Let us decompose the Weyl tensor into its electric and magnetic parts. One can define a super-Poynting vector through them via [113] $\mathcal{P}_\mu = \epsilon_{\mu\alpha\beta}\mathcal{E}^\alpha{}_\nu\mathcal{B}^{\beta\nu}$, where $\mathcal{E}_{\mu\nu} = h^\alpha{}_\mu h^\beta{}_\nu C_{\alpha\sigma\beta\gamma}t^\sigma t^\gamma$ is its electric part, $\mathcal{B}_{\mu\nu} = -\frac{1}{2}h^\alpha{}_\mu h^\beta{}_\nu \epsilon_{\alpha\sigma\gamma\kappa}C^{\gamma\kappa}{}_{\beta\rho}t^\sigma t^\rho$ is the magnetic part, t^μ is the timelike vector orthogonal to the 3-dimensional spacelike hypersurfaces, $h^\mu{}_\nu$ is the corresponding projection operator and $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor. The super-Poynting vector represents the gravitational energy flux density following its electromagnetic analogy. In [114] it is shown that choosing a transverse tetrad, rather than a principal tetrad, aligns the gravitational wave propagation direction with the super-Poynting vector. Authors indicate that if we have a device which in principal works like Szekeres' 'gravitational compass' [112] we can detect the gravitational waves locally.[1] This is of course applicable for a purely gravitational case.

4.1.2 Global approaches

For gravitational waves, Bondi mass loss [14] is one of the most widely used expression to determine the energy lost by the system via gravitational radiation at null infinity. For an asymptotically flat spacetime, with NP variables, the Bondi mass reads as [36]

$$M_B = -\frac{1}{4\pi}\int_{\mathcal{S}}\left(\psi_2^0 + \sigma^0\dot{\overline{\sigma}}^0\right)d\mathcal{S}, \tag{4.6}$$

where $\mathcal{S}$ is the closed spacelike surface located at null infinity, $\sigma = -\langle \mathbf{m}, D_{\mathbf{m}}\mathbf{l}\rangle$ is one of the NP spin coefficients and the superscript '0' represents the leading order part of the object with respect to a radial expansion. The mass loss associated with the gravitational waves is determined once the 'time' derivative, denoted by the overdot, of the Bondi mass is calculated in Bondi coordinates. Note that in the tetrad formalism approach of Bondi, the null tetrad is required to satisfy certain conditions. In the Bondi-Metzner-Sachs gauge one has

$$\kappa = \pi = \varepsilon = 0, \qquad \rho = \overline{\rho}, \qquad \tau = \overline{\alpha} + \beta, \tag{4.7}$$

[1] In fact, recently, it has been announced that the gravitational waves have been detected by local measurements of the two LIGO interferometers [115].

which gives the symmetry group of the conformal boundary at null infinity.

In terms of other global investigations, the energy loss of a relativistic body through its interaction with the external field can be traced back to Misner, Thorne and Wheeler's mass definition [116] constructed via an effective energy-momentum pseudotensor. Developed by many, including [109, 117, 118, 119], the methodology for calculation of the mass-energy loss of an isolated relativistic body via its interaction with an external field is in fact very similar to the Newtonian analysis [110].

One can calculate the mass-energy loss via [109, 110]

$$-\frac{d\mathcal{M}_{\mathcal{S}}}{dt} = \int_{\partial\mathcal{S}} (-g)\, t^{0J} n_J r^2 d\Omega, \tag{4.8}$$

where $M_{\mathcal{S}}$ is the mass inside the sphere $\mathcal{S}$ which gives the mass of the isolated object, M, to leading order under the slow rotation assumption; $\partial\mathcal{S}$ is the 2-dimensional boundary of $\mathcal{S}$, $-g$ is the square of the 4-metric density, $t^{\alpha\beta}$ is the Landau-Lifshitz pseudotensor [5], $n^J = x^J/r$ are the radial vector components and $d\Omega$ is the 2-dimensional volume element. If one keeps only the $\mathcal{E}I$ cross terms, where $\mathcal{E}_{JK} = R_{J0K0}$, $R_{\mu\nu\alpha\beta}$ is the Riemann tensor of the external field and I_{JK} is the mass quadrupole moment of the isolated body, one gets

$$-\frac{d\mathcal{M}_{\mathcal{S}}}{dt} = \frac{d}{dt}\left(\frac{1}{10}\mathcal{E}^{JK} I_{JK}\right) + \frac{1}{2}\mathcal{E}^{JK}\frac{dI_{JK}}{dt}, \tag{4.9}$$

in which only the zeroth and first order time derivatives and the leading order term in the perturbative expansion are considered. In this approach, the first term on the right hand side is interpreted as the rate of change of the interaction energy of the body and the external field, whereas the second term is interpreted as the rate of work done by the external field on the body. Therefore,

$$\frac{dW}{dt} = -\frac{1}{2}\mathcal{E}^{JK}\frac{dI_{JK}}{dt} \tag{4.10}$$

is sometimes referred to as *tidal heating* even though the energy loss/gain is not solely via the cooling/heating of the body in question [110].

There have been debates about whether or not the total mass of the body, which is taken as the sum of the self energy and the interaction energy, is ambiguous in this picture [109, 110, 111] [2]. For the time being, let us bear in mind that results obtained in this approach are true up to the leading order of the energy calculations of an external field and of an asymptotically flat spacetime which models a slowly rotating body at null infinity. Also, in general, one should be careful about using energy-momentum pseudotensors to calculate the mass-energy of a system due to the delicacies we mentioned in Chapter 2.

4.1.3 Quasilocal approaches

As mentioned previously, in Chapter 2, the effective matter plus gravitational energy, momentum and stress energy densities can be attributed to the extrinsic or intrinsic geometry of a closed, spacelike, 2-dimensional surface in many applications of general relativity. Note that these spacelike 2-surfaces can be considered as the t-constant surfaces of the (2+1) timelike boundary, $\mathcal{B}$, of the spacetime. Alternatively they can be considered as the embedded surfaces of spacelike 3-hypersurfaces or embedded surfaces of the spacetime itself [74, 120, 24, 25, 121, 26, 27].

Recall from Section 2.1.4 that Brown and York [24] define $T_B^{xy} = (\Theta\gamma^{xy} - \Theta^{xy})/(8\pi)$ as the object that carries information about the matter plus gravitational energy content of a given system by following a Hamiltonian approach. Here Θ_{xy} is the extrinsic curvature of the worldtube and γ_{xy} is the 3-metric induced on it that is fixed. Then the matter plus gravitational energy flux density, f_{BY}, follows from the worldtube derivative of the matter plus gravitational energy tensor, i.e.,

$$f_{BY} = \gamma_\mu{}^\alpha D_\alpha \left(T_B^{\mu\nu} t_\nu\right), \tag{4.11}$$

where t^μ is a timelike vector field which is not necessarily orthogonal to the t-constant spacelike surfaces $\mathcal{S}_t$, $\gamma_\mu{}^\alpha$ is the projection operator on to the worldtube and D_α is the spacetime covariant derivative.

In [122], the authors define the rate of work done on a quasilocal system via eq. (4.11) by specifically choosing t^μ not to be a timelike Killing vector field of the worldtube

[2] The discussion began with Thorne and Hartle's statement that there exists an ambiguity in the total mass-energy of the body [109]. Later, Purdue concluded that there is no ambiguity at least in the rate of work done on the system up to leading order [110]. Furthermore, Favata considered different "localisations" of gravitational energy and concluded that the total mass-energy of the system does not depend on the choice of the energy-momentum pseudotensor and is thus unambiguous [111].

metric. According to Booth and Creighton, in vacuum, the rate of work done on the system by its environment is given by

$$\frac{dW}{dt} = -\frac{1}{2}\int_{S_t} d^2x\,\sqrt{-\gamma}T_B^{\mu\nu}\$_t\gamma_{\mu\nu}, \tag{4.12}$$

where $\$_t\gamma_{\mu\nu}$ is a measure of how much the 3-boundary expands when it is perturbed along the timelike vector t^μ. Equation (4.12) is used to calculate the tidal heating quasilocally in the weak field limit, which serves as an excellent example to compare the quasilocal formalisms with the global ones. Their results show that the leading terms of the rate of work done is not exactly equal to the one given by the global method, eq. (4.10). It is only the so-called *irreversible* part, the portion that is expended to deform the body, that is equal to $\frac{1}{2}\mathcal{E}^{JK}dI_{JK}/dt$ and hence attributed to tidal heating. However, there exists an additional portion which is stored as the potential energy in the system, called the *reversible part*, which differs from the results of the global method.

In [123], Epp *et al.* take one step further and come up with a more concrete definition of matter plus gravitational energy flux between the initial, $\mathcal{S}_i$, and final, $\mathcal{S}_f$, slices of a worldtube. This approach is more concrete in the sense that the 2-surfaces have certain conditions on them. The authors define a *rigid quasilocal frame* by demanding the 2-surfaces to have zero expansion and shear when they are considered to be embedded in the worldtube. In this approach, the energy flux density in vacuum is calculated as $\alpha_\mu\mathcal{P}^\mu$. Here α_μ is the proper acceleration of the observers projected on the 2-surface, $\mathcal{P}^\mu$ are constructed via the normal and tangential projections of $T_B^{\mu\nu}$, as defined by Brown and York [24]. On the spacelike 2-surfaces $\mathcal{P}_\mu = \sigma_{\mu\nu}u_\rho T_B^{\nu\rho}$ and $\sigma_{\mu\nu}$ is the metric induced on the 2-surfaces. This is a coordinate approach. However, the conditions they impose on the spacelike 2-surface can be translated into null tetrad gauge conditions once a change of formalism is applied. In the next section, we will see that our definition of a system is not as restrictive as the one of Epp *et al.*.

4.2 Null tetrad gauge conditions and the quasilocal calculations

In the present chapter, we have no intention to discuss the advantages and disadvantages of numerical relativity calculations at finite distances[3]. However, we would like to keep track of the quasilocal observables and the null cone observables simultaneously as they are not always investigated in tandem in numerical relativity simulations.

Consider the case of a perturbed rotating black hole. In real astrophysical cases, our ultimate goal is to get information about the properties – such as the mass, angular momentum and their dissipation rates – of this black hole via the gravitational radiation we detect. In such a case, we have the freedom to choose a null tetrad for gravitational radiation calculations and a corresponding orthonormal tetrad for the quasilocal energy calculations. One of our aims, in this chapter, is to check whether or not those tetrad choices are consistent with each other when the different formalisms are considered.

For example, there is a geometrically motivated transverse tetrad, the so-called quasi-Kinnersley tetrad [127], which is considered to be one of the best choices to study the gravitational wave extraction from a perturbed Kerr black hole [128, 129, 130]. In [114], Zhang *et al.* investigate the directions of energy flow using the super-Poynting vector and show that the wave fronts of passing radiation are aligned with the quasi-Kinnersley tetrad. However, in the current section, we introduce certain null tetrad gauge conditions for a quasilocal system which are not satisfied by the quasi-Kinnersley tetrad. This might mean that even though one can measure the gravitational radiation emitted from a region properly, one might not be able to extract the quasilocal properties of its source consistently. What we mean by this sentence will be more clear once we introduce our formalism and give a detailed discussion of this specific issue in Section 4.6.

When the quasilocal properties are taken into consideration, one has to start the investigation with a proper definition of a *system*. This is the missing ingredient in many

[3]For example see Gómez and Winicour's discussion on this issue [124]. Also see [125] for a construction of a conformal method and see [126] for a pedagogical review of conformal methods in numerical relativity.

quasilocal approaches in the literature. In the present chapter, we use a rigorous geometrical method to define a generic system via the mathematical properties of our 2-surfaces $\mathbb{T}$ and $\mathbb{S}$ which we introduced in the previous chapters.

Recall that previously we considered an embedding of an oriented worldsheet with an induced metric, η_{ab}, written in terms of orthonormal basis tangent vectors, $\{E_a\}$,

$$g(E_a, E_b) = \eta_{ab}. \tag{4.13}$$

The two unit normal vectors, $\{N_i\}$, of the worldsheet were defined up to a local rotation by,

$$g(N_i, N_j) = \delta_{ij}, \tag{4.14}$$

$$g(N^i, E_a) = 0, \tag{4.15}$$

where $\{a, b\} = \{\hat{0}, \hat{1}\}$ and $\{i, j\} = \{\hat{2}, \hat{3}\}$ are the dyad indices and the Greek indices refer to 4-dimensional spacetime coordinates.

For a physically meaningful construction, we want the tangent spaces of these embedded surfaces, $\mathbb{T}$ and $\mathbb{S}$, to be integrable [32]. According to Frobenius Theorem, involutivity is a sufficient condition for the existence of an integral manifold through each point [131]. In other words, let D^k be a k-dimensional distribution on a manifold M, which is required to be C^∞. D^k is involutive if for the vector fields $\mathbf{X}, \mathbf{Y} \in D^k$ their Lie bracket satisfies $[\mathbf{X}, \mathbf{Y}] \in D^k$ [132].

Therefore our tangent basis vectors $\{E_a, N_i\}$ need to satisfy

$$[E_a, E_b] = f^c{}_{ab} E_c, \tag{4.16}$$

$$\left[N_i, N_j\right] = h^k{}_{ij} N_k. \tag{4.17}$$

Note that one can construct a complex null tetrad, $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \overline{\mathbf{m}}\}$, via an orthonormal double dyad and vice versa according to

$$E^{\mu}{}_{\hat{0}} = \frac{1}{\sqrt{2}}(l^{\mu}+n^{\mu}), \tag{4.18}$$

$$E^{\mu}{}_{\hat{1}} = \frac{1}{\sqrt{2}}(l^{\mu}-n^{\mu}), \tag{4.19}$$

$$N^{\mu}{}_{\hat{2}} = \frac{1}{\sqrt{2}}(m^{\mu}+\overline{m}^{\mu}), \tag{4.20}$$

$$N^{\mu}{}_{\hat{3}} = -\frac{i}{\sqrt{2}}(m^{\mu}-\overline{m}^{\mu}). \tag{4.21}$$

Now let us see the gauge conditions that the Frobenius theorem, when applied to the tangent spaces of $\mathbb{T}$ and $\mathbb{S}$, imposes on a null tetrad constructed via the tangent vectors of $\mathbb{T}$ and $\mathbb{S}$. We can rewrite eq. (4.16) as

$$E^{\mu}{}_{a}D_{\mu}E^{\nu}{}_{b} - E^{\mu}{}_{b}D_{\mu}E^{\nu}{}_{a} = f^{c}{}_{ab}\,E^{\nu}{}_{c} := F^{\nu}{}_{ab}. \tag{4.22}$$

Considering the only non zero component of F_{ab}, i.e., $F_{\hat{0}\hat{1}} = -F_{\hat{1}\hat{0}}$ and expressions (4.18)-(4.19) we can write

$$\begin{aligned} F^{\nu}{}_{\hat{0}\hat{1}} &= E^{\mu}{}_{\hat{0}}D_{\mu}E^{\nu}{}_{\hat{1}} - E^{\mu}{}_{\hat{1}}D_{\mu}E^{\nu}{}_{\hat{0}} = f^{\hat{0}}{}_{\hat{0}\hat{1}}\,E^{\nu}{}_{\hat{0}} + f^{\hat{1}}{}_{\hat{0}\hat{1}}\,E^{\nu}{}_{\hat{1}} \\ &= \frac{1}{2}\left[(l^{\mu}+n^{\mu})D_{\mu}(l^{\nu}-n^{\nu}) - (l^{\mu}-n^{\mu})D_{\mu}(l^{\nu}+n^{\nu})\right] \\ &= \frac{1}{\sqrt{2}}\left[f^{\hat{0}}{}_{\hat{0}\hat{1}}\,(l^{\nu}+n^{\nu}) + f^{\hat{1}}{}_{\hat{0}\hat{1}}\,(l^{\nu}-n^{\nu})\right]. \end{aligned} \tag{4.23}$$

Thus,

$$(D_{\mathbf{l}}n^{\nu} - D_{\mathbf{n}}l^{\nu}) = -\frac{1}{\sqrt{2}}\left[\left(f^{\hat{0}}{}_{\hat{0}\hat{1}} + f^{\hat{1}}{}_{\hat{0}\hat{1}}\right)l^{\nu} + \left(f^{\hat{0}}{}_{\hat{0}\hat{1}} - f^{\hat{1}}{}_{\hat{0}\hat{1}}\right)n^{\nu}\right]. \tag{4.24}$$

Now if we take the inner product of both sides of eq. (4.24) with the null vector **m** we get

$$\langle \mathbf{m}, D_{\mathbf{l}}\mathbf{n}\rangle - \langle \mathbf{m}, D_{\mathbf{n}}\mathbf{l}\rangle = \overline{\pi} - (-\tau) = 0, \tag{4.25}$$

which follows from the propagation equations (A.10) and (A.12) of the spin coefficients of the Newman-Penrose formalism [34].

Likewise when we rewrite eq. (4.17) we get

$$N^{\mu}{}_{i}D_{\mu}N^{\nu}{}_{j} - N^{\mu}{}_{j}D_{\mu}N^{\nu}{}_{i} = h^{k}{}_{ij}N^{\nu}{}_{k} := H^{\nu}{}_{ij}. \tag{4.26}$$

If we consider the non-vanishing component $H_{\hat{2}\hat{3}}$ with the expressions (4.20)-(4.21) we can write

$$\begin{aligned}
H^{\nu}{}_{\hat{2}\hat{3}} &= N^{\mu}{}_{\hat{2}} D_{\mu} N^{\nu}{}_{\hat{3}} - N^{\mu}{}_{\hat{3}} D_{\mu} N^{\nu}{}_{\hat{2}} = h^{\hat{2}}{}_{\hat{2}\hat{3}} N^{\nu}{}_{\hat{2}} + h^{\hat{3}}{}_{\hat{2}\hat{3}} N^{\nu}{}_{\hat{3}} \\
&= -\frac{i}{2}(m^{\mu}+\overline{m}^{\mu}) D_{\mu}(m^{\nu}-\overline{m}^{\nu}) + \frac{i}{2}(m^{\mu}-\overline{m}^{\mu}) D_{\mu}(m^{\nu}+\overline{m}^{\nu}) \\
&= \frac{1}{\sqrt{2}}\left[h^{\hat{2}}{}_{\hat{2}\hat{3}}(m^{\nu}+\overline{m}^{\nu}) - i h^{\hat{3}}{}_{\hat{2}\hat{3}}(m^{\nu}-\overline{m}^{\nu})\right].
\end{aligned} \tag{4.27}$$

Hence,

$$(D_{\mathbf{m}}\overline{m}^{\nu} - D_{\overline{\mathbf{m}}} m^{\nu}) = -\frac{1}{\sqrt{2}}\left[m^{\nu}\left(h^{\hat{3}}{}_{\hat{2}\hat{3}} + i h^{\hat{2}}{}_{\hat{2}\hat{3}}\right) - \overline{m}^{\nu}\left(h^{\hat{3}}{}_{\hat{2}\hat{3}} - i h^{\hat{2}}{}_{\hat{2}\hat{3}}\right)\right]. \tag{4.28}$$

Taking the inner product of both sides of eq. (4.28) with the null vectors **l** and **n** respectively gives,

$$\langle \mathbf{l}, D_{\mathbf{m}}\overline{\mathbf{m}}\rangle - \langle \mathbf{l}, D_{\overline{\mathbf{m}}}\mathbf{m}\rangle = \overline{\rho} - \rho = 0, \tag{4.29}$$

$$\langle \mathbf{n}, D_{\mathbf{m}}\overline{\mathbf{m}}\rangle - \langle \mathbf{n}, D_{\overline{\mathbf{m}}}\mathbf{m}\rangle = (-\mu) - (-\overline{\mu}) = 0, \tag{4.30}$$

which follow from the propagation equation (A.18).

Therefore we will state that for quasilocal energy calculations in our 2+2 approach, the following three null gauge conditions must be satisfied,

$$\tau + \overline{\pi} = 0, \qquad \rho = \overline{\rho}, \qquad \mu = \overline{\mu}. \tag{4.31}$$

It is easy to check that under a Type-III Lorentz transformation of the complex null tetrad, i.e.,

$$\mathbf{l} \to a^2 \mathbf{l}, \tag{4.32}$$

$$\mathbf{n} \to \frac{1}{a^2}\mathbf{n}, \tag{4.33}$$

$$\mathbf{m} \to e^{2i\theta}\mathbf{m}, \tag{4.34}$$

$$\overline{\mathbf{m}} \to e^{-2i\theta}\overline{\mathbf{m}}, \tag{4.35}$$

the gauge conditions (4.31) are preserved. This is because transformation of the spin coefficients τ, π, ρ, μ under Type-III Lorentz transformation follows as [133]

$$\tau \rightarrow e^{2i\theta}\tau, \tag{4.36}$$
$$\pi \rightarrow e^{-2i\theta}\pi, \tag{4.37}$$
$$\rho \rightarrow a^2\rho, \tag{4.38}$$
$$\mu \rightarrow \frac{1}{a^2}\mu, \tag{4.39}$$

in which a^2 and 2θ respectively refer to the *boost* and *spin* parameters in Newman-Penrose formalism. They are both real constants. Note that this transformation corresponds to

$$E^{\mu}{}_{\hat{0}} \rightarrow \gamma\left(E^{\mu}{}_{\hat{0}} - \beta E^{\mu}{}_{\hat{1}}\right), \tag{4.40}$$
$$E^{\mu}{}_{\hat{1}} \rightarrow \gamma\left(E^{\mu}{}_{\hat{1}} - \beta E^{\mu}{}_{\hat{0}}\right), \tag{4.41}$$

where

$$\beta = \frac{a^4-1}{a^4+1} \qquad and \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \tag{4.42}$$

meaning that a Type-III Lorentz transformation of the null tetrad corresponds to the boosting of the timelike observers along $E^{\mu}{}_{\hat{1}}$ on $\mathbb{T}$. This is the property we want to preserve in the definition and the investigation of our quasilocal system.

4.3 Raychaudhuri equation with the Newman-Penrose formalism

We use the relations (4.18)-(4.21) in order to rewrite the contracted Raychaudhuri equation of our 2-dimensional timelike worldsheet, eq. (2.111), in the language of the NP formalism. This will allow us to compare the results of the investigations of the energy exchange mechanisms built on null cone variables and the notation that is used in quasilocal energy calculations.

Note that eq. (2.111) is built on the extrinsic geometry of $\mathbb{T}$ and $\mathbb{S}$. Those extrinsic objects, like curvature, rotation and twist, are all measures of how much the dyad vectors change when they are propagated along each other. Likewise in the NP formalism, spin coefficients are defined via the changes of null vectors when they are propagated along each other with the relevant projections. A short summary of the NP formalism and the detailed calculations of our formalism transformation can be found in Appendices A and B respectively.

When the formalism transformation is applied, the contracted Raychaudhuri equation, (2.111), of $\mathbb{T}$ can be conveniently written as

$$\tilde{\nabla}_{\mathbb{T}}\mathcal{J} = -\tilde{\nabla}_{\mathbb{S}}\mathcal{K} - \mathcal{J}^2 - \mathcal{K}^2 + \mathcal{R}_{\mathcal{W}}, \tag{4.43}$$

where

$$\begin{aligned}
\tilde{\nabla}_{\mathbb{T}}\mathcal{J} := \eta^{ab}\delta^{ij}\tilde{\nabla}_b J_{aij} &= [D_{\mathbf{n}}(\rho+\bar{\rho}) - D_{\mathbf{l}}(\mu+\bar{\mu})] \qquad (4.44)\\
&- [(\varepsilon+\bar{\varepsilon})(\mu+\bar{\mu}) + (\gamma+\bar{\gamma})(\rho+\bar{\rho})]\\
&+ 2[(\varepsilon-\bar{\varepsilon})(\mu-\bar{\mu}) + (\gamma-\bar{\gamma})(\rho-\bar{\rho})],\\
\tilde{\nabla}_{\mathbb{S}}\mathcal{K} := \eta^{ab}\delta^{ij}\tilde{\nabla}_i K_{abj} &= D_{\mathbf{m}}(\pi-\bar{\tau}) + D_{\overline{\mathbf{m}}}(\bar{\pi}-\tau) \qquad (4.45)\\
&- \left[(\bar{\alpha}-\beta)(\pi-\bar{\tau}) + \left(\alpha-\bar{\beta}\right)(\bar{\pi}-\tau)\right]\\
&+ 2\left[(\bar{\alpha}+\beta)(\pi+\bar{\tau}) + \left(\alpha+\bar{\beta}\right)(\bar{\pi}+\tau)\right],\\
\mathcal{J}^2 := J_{bik}J_{alj}\eta^{ab}\delta^{ij}\delta^{lk} &= 2\left(\mu\bar{\rho}+\bar{\mu}\rho+\sigma\lambda+\bar{\sigma}\bar{\lambda}\right), \qquad (4.46)\\
\mathcal{K}^2 := K_{bci}K_{adj}\eta^{ab}\eta^{cd}\delta^{ij} &= -2(\kappa\nu+\overline{\kappa\nu}+\pi\tau+\overline{\pi\tau}), \qquad (4.47)\\
\mathcal{R}_{\mathcal{W}} := g(R(E_b,N_i)E_a,N_j)\eta^{ab}\delta^{ij} &= D_{\mathbf{n}}(\rho+\bar{\rho}) - D_{\mathbf{l}}(\mu+\bar{\mu}) + D_{\mathbf{m}}(\pi-\bar{\tau}) + D_{\overline{\mathbf{m}}}(\bar{\pi}-\tau)\\
&- \left[\left(\alpha-\bar{\beta}\right)(\bar{\pi}-\tau) + (\bar{\alpha}-\beta)(\pi-\bar{\tau})\right]\\
&- [(\varepsilon+\bar{\varepsilon})(\mu+\bar{\mu}) + (\gamma+\bar{\gamma})(\rho+\bar{\rho})] - 2(\kappa\nu+\overline{\kappa\nu})\\
&+ 2\left(\rho\bar{\mu}+\bar{\rho}\mu+\lambda\sigma+\bar{\lambda}\bar{\sigma}\right). \qquad (4.48)
\end{aligned}$$

An alternative, more compact expression for $\mathcal{R}_{\mathcal{W}}$ is

$$\mathcal{R}_{\mathcal{W}} = -2\left(\psi_2+\bar{\psi}_2+4\Lambda\right). \tag{4.49}$$

Now if we substitute the terms (4.44)-(4.49) back into eq. (4.43) we see that the Raychaudhuri equation is not yet satisfied. This is simply because Capovilla and Guven impose the integrability condition in their formalism to define the extrinsic objects[4] and we did not impose it after our change of formalism. We must further impose the null tetrad gauge conditions introduced in Section 4.2. Thus, with $\tau+\bar{\pi}=0$, $\rho=\bar{\rho}$ and $\mu=\bar{\mu}$

[4]This can be seen by checking the symmetries of the extrinsic objects introduced at the previous section.

we get

$$
\begin{aligned}
\tilde{\nabla}_{\mathbb{T}}\mathcal{J} &= 2\left(D_{\mathbf{n}}\rho - D_{\mathbf{l}}\mu\right) - 2\left[(\varepsilon+\bar{\varepsilon})\mu + (\gamma+\bar{\gamma})\rho\right], &(4.50)\\
\tilde{\nabla}_{\mathbb{S}}\mathcal{K} &= 2\left(D_{\mathbf{m}}\pi - D_{\overline{\mathbf{m}}}\tau\right) - 2\left[(\bar{\alpha}-\beta)\pi + \left(\alpha-\bar{\beta}\right)\bar{\pi}\right], &(4.51)\\
\mathcal{J}^2 &= 4\mu\rho + 2\left(\sigma\lambda + \bar{\sigma}\bar{\lambda}\right), &(4.52)\\
\mathcal{K}^2 &= -2\left(\kappa\nu + \overline{\kappa\nu}\right) + 2\left(\pi\bar{\pi} + \tau\bar{\tau}\right), &(4.53)\\
\mathcal{R}_{\mathcal{W}} &= 2\left[D_{\mathbf{n}}\rho - D_{\mathbf{l}}\mu\right] + 2\left[D_{\mathbf{m}}\pi - D_{\overline{\mathbf{m}}}\tau\right]\\
&- 2\left[(\bar{\alpha}-\beta)\pi + \left(\alpha-\bar{\beta}\right)\bar{\pi}\right] - 2\left[(\varepsilon+\bar{\varepsilon})\mu + (\gamma+\bar{\gamma})\rho\right]\\
&- 2\left(\kappa\nu + \overline{\kappa\nu}\right) + 2\left(\tau\bar{\tau} + \pi\bar{\pi}\right) + 4\mu\rho + 2\left(\sigma\lambda + \bar{\sigma}\bar{\lambda}\right), &(4.54)
\end{aligned}
$$

and the alternative expression (4.49) is unchanged. These variables now satisfy the Raychaudhuri equation as expected.

We further note that since the Einstein field equations have not yet been applied, (4.50)-(4.54) are purely geometrical results irrespective of the underlying gravitational theory that governs the dynamics of the quasilocal observers. In order to satisfy the Einstein equations, all 16 of the field equations of the spin coefficients should be satisfied. However, we need to emphasise that this version of the contracted Raychaudhuri equation contains all the information contained in two of the NP spin field equations. Let us consider the following NP spin field equations

$$D_{\mathbf{l}}\mu - D_{\mathbf{m}}\pi = \mu\bar{\rho} - (\varepsilon+\bar{\varepsilon})\mu + \sigma\lambda + \pi\bar{\pi} - (\bar{\alpha}-\beta)\pi - \kappa\nu + \psi_2 + 2\Lambda, \tag{4.55}$$

$$D_{\mathbf{n}}\rho - D_{\overline{\mathbf{m}}}\tau = -\bar{\mu}\rho + (\gamma+\bar{\gamma})\rho - \sigma\lambda - \tau\bar{\tau} - \left(\alpha-\bar{\beta}\right)\tau + \kappa\nu - \psi_2 - 2\Lambda. \tag{4.56}$$

If we take $(4.55)+(4.55)^{*}-(4.56)-(4.56)^{*}$, where $*$ denotes the complex conjugate, then the result is the contracted Raychaudhuri equation of the worldsheet under our gauge conditions. We will not attempt to restrict the general set of equations (4.50)-(4.54) by further imposing the Einstein equations. Rather, we will apply it to spacetimes that are already solutions of the Einstein field equations.

4.4 A work-energy relation

In this section we are going to define quasilocal charges by using the terms that appear in the Raychaudhuri equation. Ultimately we will make definitions so as to end up with a work-energy relation that looks like the following

$$E_{\text{Total}} = E_{\text{Dilatational}} + E_{\text{Rotational}} + W_{\text{Tidal}}. \tag{4.57}$$

In doing so, one of Kijowski's quasilocal energy definitions will be our anchor. Let us recall the two energy definitions made by Kijowski which are derived from a gravitational action [25],

$$E_{\text{K1}} = -\frac{1}{16\pi}\oint_{\mathbb{S}} d\mathbb{S}\left[\frac{H-k_0^2}{k_0}\right], \tag{4.58}$$

$$E_{\text{K2}} = -\frac{1}{8\pi}\oint_{\mathbb{S}} d\mathbb{S}\left[\sqrt{H}-k_0\right], \tag{4.59}$$

Previously, in Chapter 3, we identified eq. (4.58) as internal energy since it was associated with the quasilocal energy of a system in equilibrium which can potentially be used to do work, dissipate heat or exchange energy in other forms. The second expression (4.59) is usually interpreted as the invariant mass energy of the system that is an analogue of a proper mass of a particle [26]. Therefore if we are after an expression which represents the energy that can be exchanged by the system, H, should be our central object.

The quasilocal energy definitions E_{K1} and E_{K2} of Kijowski both have the functional form H^p with $p = 1$ and $p = 1/2$ respectively. This is due to Kijowski applying a Legendre transform on the boundary Hamiltonian with different boundary conditions as mentioned in Chapter 2. However, this does not cause any problem in terms of the dimensionality of the quasilocal energies as the so-called reference terms, which make sure that the energy definitions are boost invariant, do not appear in the same format.

In Chapter 3, we defined quasilocal thermodynamic potentials at equilibrium for spherically symmetric spacetimes by using the terms that appear in the contracted Raychaudhuri equation, (2.111), of $\mathbb{T}$. We applied our formalism for metrics with boundary conditions $g_{00} = 1$, $g_{0A} = 0$ when the quasilocal observers are located at the apparent horizon[5]. Therefore the quasilocal charges defined in [33] take the same form as E_{K2}. Note that this refers to a very special state of the system in question.

In the present chapter, we would like to define quasilocal charges for nonequilibrium

[5]For the Schwarzschild case, this corresponds to quasilocal observers at event horizon in which $g_{00} \to 0$ at the quasilocal thermodynamic equilibrium defined in Chapter 3.

states and we would like to go beyond spherical symmetry. We will consider space-times with metrics that have time independent components for the induced 2-metric on $\mathbb{S}$ just as Kijowski did to define E_{K1}. In order to define the quasilocal charges we will first multiply the contracted Raychaudhuri equation (4.43) by two[6], and add reference energy term, k_0^2 term to each side. Since all of the terms that appear in eq. (4.43) have dimension $(length)^{-2}$ on account of their relationship to the Riemann tensor, to obtain a quasilocal energy expression we further divide by k_0 before integrating the equation on our closed 2-surface $\mathbb{S}$. Then we obtain the following quasilocal charges

$$\begin{aligned}
E_{\mathrm{Tot}} &= -\frac{1}{16\pi}\oint_{\mathbb{S}} d\mathbb{S}\left[\frac{-\left(2\tilde{\nabla}_{\mathbb{T}}\mathcal{J}+k_0^2\right)}{k_0}\right], && (4.60)\\
E_{\mathrm{Dil}} &= -\frac{1}{16\pi}\oint_{\mathbb{S}} d\mathbb{S}\left[\frac{2\mathcal{J}^2-k_0^2}{k_0}\right], && (4.61)\\
E_{\mathrm{Rot}} &= -\frac{1}{16\pi}\oint_{\mathbb{S}} d\mathbb{S}\left[\frac{2\tilde{\nabla}_{\mathbb{S}}\mathcal{K}+2\mathcal{K}^2}{k_0}\right], && (4.62)\\
W_{\mathrm{Tid}} &= -\frac{1}{16\pi}\oint_{\mathbb{S}} d\mathbb{S}\left[\frac{-2\mathcal{R}_{\mathcal{W}}}{k_0}\right], && (4.63)
\end{aligned}$$

so that

$$E_{\mathrm{Tot}} = E_{\mathrm{Dil}}+E_{\mathrm{Rot}}+W_{\mathrm{Tid}} \quad (4.64)$$

is satisfied.

In the following sections, we will discuss our reasons for these quasilocal charge definitions. The reasons behind naming our quasilocal charges like energy associated with dilatational or rotational degrees of freedom and work done by tidal fields of the system will be explained.

4.4.1 Energy associated with dilatational degrees of freedom

In spherical symmetry, we were able to write $\mathcal{J}^2 := J_{bik}J_{alj}\eta^{ab}\delta^{ij}\delta^{lk}$ in terms of the mean extrinsic curvature, H, of $\mathbb{S}$ via $2\mathcal{J}^2 = H$. Note that confining the quasilocal observers to radial world lines in a spherically symmetric system results in corresponding, purely radial, null congruences that are shear-free. Indeed, for the generic

[6]The reason behind this factor of 2 will be more clear in the following sections.

case,

$$H := J_{aik}J_{bjl}\eta^{ab}\delta^{ik}\delta^{jl} = 2(\rho+\bar{\rho})(\mu+\bar{\mu}), \tag{4.65}$$
$$\mathcal{J}^2 := J_{ail}J_{bjk}\eta^{ab}\delta^{ik}\delta^{jl} = 2\left(\mu\bar{\rho}+\bar{\mu}\rho+\sigma\lambda+\bar{\sigma}\bar{\lambda}\right). \tag{4.66}$$

Therefore with two of our null tetrad gauge conditions, $\rho=\bar{\rho}$, $\mu=\bar{\mu}$ and the shear-free case, $\sigma=0$,

$$H = 2\mathcal{J}^2 = 4\left(\mu\bar{\rho}+\bar{\mu}\rho+\sigma\lambda+\bar{\sigma}\bar{\lambda}\right) = 8\mu\rho. \tag{4.67}$$

This is natural for radially moving observers of spherically symmetric systems. However, it is not clear which of the terms in (4.65) and (4.66) carries more information about the generic system in question.

According to the Goldberg-Sachs theorem, there exists a shear-free null congruence, k^μ, for a vacuum spacetime if [12]

$$k_{[\mu}C_{\nu]\alpha\beta[\gamma}k_{\sigma]}k^\alpha k^\beta = 0 \tag{4.68}$$

is satisfied. This means that if we wish to have the shear-free property, we need to pick a principal null tetrad for our systems in vacuum. However, there is no such $a\,priori$ necessity for our formalism to hold.

In [134], Adamo *et al.* investigate the shear free null geodesics of asymptotically flat spacetimes in detail. They note that the shear-free or asymptotically shear-free null congruences may provide information about the asymptotic center of mass or intrinsic magnetic dipole in certain cases. Also the importance of the twistor theory, which is solely constructed on shear-free null congruences, cannot be denied. At this point, we should also emphasise that the spacetimes we are interested in are not necessarily asymptotically flat.

In [135], Ellis investigated shear-free timelike and null congruences. He concluded that by imposing a shear-free condition on the null congruences, one puts a restriction on the way the distant matter can influence the local gravitational field. In that case, there is an information loss. Note that shear is also the central concept of Bondi's mass loss formulation. It is only if the null congruence has shear, that one can define a $news\,function$ which is solely responsible for the mass loss via gravitational radiation at null infinity [14]. Ellis also emphasised the fact that, a nonrotating expanding null geodesics congruence in vacuum must have shear. Thus we cannot completely separate the effect of dilatation and shear for null congruences. We will combine them in the quasilocal charge constructed from the $\mathcal{J}^2$ term, (4.52), and write

$$\begin{aligned} E_{\mathrm{Dil}} &= -\frac{1}{16\pi}\oint_{\mathbb{S}} d\mathbb{S}\left[\frac{2\mathcal{J}^2 - k_0^2}{k_0}\right] \\ &= -\frac{1}{16\pi}\oint_{\mathbb{S}} d\mathbb{S}\left[\frac{8\mu\rho + 4\left(\sigma\lambda + \overline{\sigma}\overline{\lambda}\right) - k_0^2}{k_0}\right]. \end{aligned} \tag{4.69}$$

Since we claim that the Raychaudhuri equation of the worldsheet incorporates the physically meaningful quasilocal energy densities, one might ask what is the direct connection of our $\mathcal{J}^2$ term (4.66) to the boundary Hamiltonian which is generically written in terms of the mean extrinsic curvature H, (4.65). The link lies in the Gauss equation of the 2-surface $\mathbb{S}$ when it is embedded directly into spacetime [136], i.e.,

$$g(R(N_k, N_l)N_j, N_i) = \mathcal{R}_{ijkl} - J_{aik}J_{bjl}\eta^{ab} + J_{ajk}J_{bil}\eta^{ab}, \tag{4.70}$$

where $\mathcal{R}_{ijkl}$ is the Riemann tensor associated with the 2-dimensional metric induced on $\mathbb{S}$. If we contract eq. (4.70) with $\delta^{ik}\delta^{jl}$ we find

$$\mathcal{J}^2 = H - \mathcal{R}_{\mathbb{S}} + 2\left(\Psi_2 + \overline{\Psi_2} - 2\Lambda - 2\Phi_{11}\right), \tag{4.71}$$

in which $\mathcal{R}_{\mathbb{S}} := \mathcal{R}_{ijkl}\delta^{ik}\delta^{jl}$ is the scalar intrinsic curvature of $\mathbb{S}$ and the derivation of $g(R(N_k, N_l)N^l, N^k) = -2\left(\Psi_2 + \overline{\Psi_2} - 2\Lambda - 2\Phi_{11}\right)$ can be found in Appendix C. Equation (4.71) not only allows us to connect our $\mathcal{J}^2$ term to the boundary Hamiltonian of general relativity, but it can also be used to relate different quasilocal energy definitions which are built on either the extrinsic or intrinsic curvature of $\mathbb{S}$.

4.4.2 Energy associated with rotational degrees of freedom

In the previous subsection we defined the quasilocal energy associated with the dilatational degrees of freedom by combining the real divergence and the possibly existing shear of the null congruence that spans on the timelike surface $\mathbb{T}$. Now we will distinguish which spin coefficients are most significant in defining the energy associated with the rotational degrees of freedom.

Recall that by imposing the integrability conditions on our local dyad we made sure that the tangent vectors of the spacelike surface $\mathbb{S}$ always stay within the surface. Later, we transformed our construction into the NP formalism and stated that these conditions imply that the the null vectors $\{\mathbf{m}, \overline{\mathbf{m}}\}$, constructed from the spacelike dyad of $\mathbb{S}$, should span on $\mathbb{S}$ throughout the evolution of the quasilocal system. Then the magnitude of the change of these null vectors should be related to how much the quasilocal system rotates. Note that this interpretation makes sense only when one forces $\{\mathbf{m}, \overline{\mathbf{m}}\}$ to stay on $\mathbb{S}$ throughout the evolution.

Now let us define the spacetime covariant derivative via the directional covariant derivatives of the null tetrad and write

$$D_\mu = -l_\mu D_{\mathbf{n}} - n_\mu D_{\mathbf{l}} + m_\mu D_{\overline{\mathbf{m}}} + \overline{m}_\mu D_{\mathbf{m}}. \tag{4.72}$$

Then the change in components of $\{\mathbf{m}, \overline{\mathbf{m}}\}$ follow as

$$\begin{aligned} D_\mu m^\mu &= -\langle \mathbf{l}, D_{\mathbf{n}}\mathbf{m}\rangle - \langle \mathbf{n}, D_{\mathbf{l}}\mathbf{m}\rangle + \langle \mathbf{m}, D_{\overline{\mathbf{m}}}\mathbf{m}\rangle + \langle \overline{\mathbf{m}}, D_{\mathbf{m}}\mathbf{m}\rangle, \\ D_\mu \overline{m}^\mu &= -\langle \mathbf{l}, D_{\mathbf{n}}\overline{\mathbf{m}}\rangle - \langle \mathbf{n}, D_{\mathbf{l}}\overline{\mathbf{m}}\rangle + \langle \mathbf{m}, D_{\overline{\mathbf{m}}}\overline{\mathbf{m}}\rangle + \langle \overline{\mathbf{m}}, D_{\mathbf{m}}\overline{\mathbf{m}}\rangle, \end{aligned}$$

By using equations (A.15)-(A.18) we get

$$\begin{aligned} D_\mu m^\mu &= (\overline{\pi} - \tau) + (\beta - \overline{\alpha}), \\ D_\mu \overline{m}^\mu &= (\pi - \overline{\tau}) + \left(\overline{\beta} - \alpha\right). \end{aligned}$$

Therefore, the spin coefficients $\{\pi,\ \tau,\ \alpha,\ \beta\}$, their complex conjugates and their changes when one perturbs them on $\mathbb{S}$ can be used to define the energy associated with the rotational degrees of freedom. Since the terms $\tilde{\nabla}_{\mathbb{S}}\mathcal{K}$, (4.51), and $\mathcal{K}^2$, (4.53), involve these spin coefficients and their changes we define

$$\begin{aligned} E_{\text{Rot}} &= -\frac{1}{16\pi}\oint_{\mathbb{S}} d\mathbb{S}\left[\frac{2\tilde{\nabla}_{\mathbb{S}}\mathcal{K} + 2\mathcal{K}^2}{k_0}\right] \\ &= -\frac{1}{16\pi}\oint_{\mathbb{S}} d\mathbb{S}\frac{4\left[D_{\mathbf{m}}\pi - D_{\overline{\mathbf{m}}}\tau - \pi(\overline{\alpha} - \beta) - \overline{\pi}\left(\alpha - \overline{\beta}\right) + \pi\overline{\pi} + \tau\overline{\tau} - \kappa\nu - \overline{\kappa\nu}\right]}{k_0} \end{aligned} \tag{4.73}$$

Note that the term $(\kappa\nu + \overline{\kappa\nu})$ vanishes if one picks the null vector $\mathbf{l}$ or $\mathbf{n}$ that spans on $\mathbb{T}$ to be a geodesic, i.e., $\kappa = 0$ or $\nu = 0$. In that case E_{Rot} can be written purely in terms of the spin coefficients $\{\pi,\ \tau,\ \alpha,\ \beta\}$. However, there is no geometric or physical reason for us to demand our null congruences to be geodesic, and we will not impose the geodesic condition for the time being.

4.4.3 Work done by tidal distortions

If we want to understand the properties of a system via its energy exchange mechanisms we need to account for the different types of associated energies, especially in the nonequilibrium case. One needs to be careful about what is actually measured by the quasilocal observers. What is physical for any one observer is the tidal acceleration as measured by that observer's local ruler and clock. The work done by tidal distortions of the the whole system, however, requires the quasilocal observers to be placed in such a geometric configuration that the observers all agree on the fact that they are measuring the properties of the same system. In the previous sections, we stated that this is guaranteed by our integrability conditions.

In [137], Hartle investigates the changes in the shape of an instantaneous horizon of a rotating black hole through the intrinsic scalar curvature, $\mathcal{R}_{\mathbb{S}}$, of a spacelike 2-surface when it is embedded into a 4-dimensional spacetime. He $chooses$ a null tetrad gauge $so\ that$ $\mathcal{R}_{\mathbb{S}}$ can be written in terms of a simple combination of Ψ_2 and the spin coefficients in vacuum. In the end, he finds $\mathcal{R}_{\mathbb{S}} = 4Re(-\Psi_2 + \rho\mu - \lambda\sigma)$. In [16], Hayward provide a quasilocal version of the Bondi-Sach mass via the Hawking mass [74], in which the central object is again the complex intrinsic scalar curvature given by $\mathcal{R}^H_{\mathbb{S}} = -\Psi_2 + \sigma\sigma' - \rho\rho' + \Phi_{11} + \Pi$, in the formalism of weighted spin coefficients.

We believe that the $\mathcal{R}_{\mathcal{W}}$ term that appears in eq. (4.54) has a more fundamental meaning than $\mathcal{R}_{\mathbb{S}}$ in terms of the tidal distortion. Recall that in Chapter 3, we defined a *relative work density* term, that mimics $W = \vec{F} \cdot \vec{x}$ by

$$\left(\frac{d^2\xi^\mu}{d\tau^2}\right)\xi_\mu = R^\gamma{}_{\nu\rho\sigma}u^\nu u^\rho \xi^\sigma \xi_\gamma, \tag{4.74}$$

by considering the geodesic deviation equation. We noted that, in the 3+1 picture, connecting the two worldlines is essentially nonlocal. The reason for applying the geodesic deviation equation only for neighbouring worldlines is due to the fact that the observers are trying to approximate the value of a quantity, which is essentially quasilocal, locally. Therefore the quantity (4.74) in the 2+2 picture, i.e., $\mathcal{R}_{\mathcal{W}} = g(R(E_b, N_i)E_a, N_j)\eta^{ab}\delta^{ij} = -2\left(\psi_2 + \overline{\psi}_2 + 4\Lambda\right)$, should have a more fundamental importance, as it is an intrinsically quasilocal quantity. Therefore by eq. (4.49)

$$\begin{aligned} W_{\text{Tid}} &= -\frac{1}{16\pi}\oint_{\mathbb{S}} d\mathbb{S}\left[\frac{-2\mathcal{R}_{\mathcal{W}}}{k_0}\right] \\ &= -\frac{1}{16\pi}\oint_{\mathbb{S}} d\mathbb{S}\left[\frac{4\left(\psi_2+\bar{\psi}_2+4\Lambda\right)}{k_0}\right]. \end{aligned} \tag{4.75}$$

Note that the quasilocal tidal work of the system is written purely in terms of the Coulomb-like Weyl curvature scalar, ψ_2, and the Ricci scalar of the spacetime due to $\Lambda = R/24$. This interpretation does not contradict our intuition, since one would expect the quasilocal observers to measure greater magnitude of tidal distortion under higher Coulomb-like attraction and a higher Ricci curvature.

4.4.4 Total energy

In Chapter 3 we associated the $\sqrt{2\mathcal{J}^2}$ term with the Helmholtz free energy density for spherically symmetric systems in equilibrium. Likewise $\sqrt{2\left|\tilde{\nabla}_{\mathbb{T}}\mathcal{J}\right|}$ was interpreted as the Gibbs free energy density of the system that *includes* the energy that is spontaneously exchanged with the surroundings to relax the system into its current state. However, in the present chapter, we do not attempt to give a thermodynamic interpretation to the Raychaudhuri equation of Capovilla and Guven since systems far from equilibrium cannot be assigned unique thermodynamic relations even in classical thermodynamics [70]. Therefore, by using the term $\tilde{\nabla}_{\mathbb{T}}\mathcal{J}$, (4.50), the total energy is represented by

$$\begin{aligned} E_{\text{Tot}} &= -\frac{1}{16\pi}\oint_{\mathbb{S}} d\mathbb{S}\left[\frac{-\left(2\tilde{\nabla}_{\mathbb{T}}\mathcal{J}+k_0^2\right)}{k_0}\right] \\ &= -\frac{1}{16\pi}\oint_{\mathbb{S}} d\mathbb{S}\left[\frac{-\left(4\left[D_{\mathbf{n}}\rho - D_{\mathbf{l}}\mu\right] - 4\left[(\varepsilon+\bar{\varepsilon})\mu + (\gamma+\bar{\gamma})\rho\right] + k_0^2\right)}{k_0}\right]. \end{aligned} \tag{4.76}$$

Here the total energy combines two types of terms: (i) the quasilocal energy the system possesses, (ii) the energy that is expended by the ‘internal’(tidal) forces to bring the quasilocal observers in a geometric configuration to define $\mathbb{S}$. The first piece further splits into the energy associated with dilatational and rotational degrees of freedom. The second piece can be viewed as the energy that has already been expended by the system in order for it to create ‘room’ for itself.

4.4.5 On the boost invariance of the quasilocal charges

Previously, in Section 4.2, it was shown that our tetrad conditions, (4.31), are invariant under Type-III Lorentz transformations which correspond to the boosting of physical observers in the only spacelike direction, $E^{\mu}{}_{\hat{1}}$, defined on $\mathbb{T}$. We also stated that for a well defined construction, one would expect the matter plus gravitational energy of the system to be boost invariant.

In Appendix C.2 we show that all of the terms, (4.50)-(4.54), that appear in the contracted Raychaudhuri equation are invariant under such spin-boost transformations. Therefore all of the quasilocal charges we defined in the current section are invariant under the boosting of the observers along the spacelike direction orthogonal to $\mathbb{S}$.

4.5 Applications

4.5.1 Radiating Vaidya spacetime

The Vaidya spacetime is used in investigations of radiating stars. It is associated with a spherically symmetric metric which reduces to the Schwarzschild metric when the mass function of the body is taken to be a constant. In standard coordinates with null coordinate, u, Vaidya metric is

$$ds^2 = -\left(1 - \frac{2M(u)}{r}\right) du^2 - 2du\,dr + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{4.77}$$

Let us pick the following complex null tetrad, $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \overline{\mathbf{m}}\}$, with

$$l^\mu = \partial_u - \left(\frac{1}{2} - \frac{M(u)}{r}\right)\partial_r, \tag{4.78}$$

$$n^\mu = \partial_r, \tag{4.79}$$

$$m^\mu = \frac{1}{\sqrt{2}}\left(\frac{1}{r}\partial_\theta + \frac{i}{r\sin\theta}\partial_\phi\right). \tag{4.80}$$

For such a complex null tetrad, κ, ν, σ, λ, τ, π all vanish so that $\pi + \overline{\tau} = 0$ is trivially satisfied. Also $\rho = \overline{\rho}$, $\mu = \overline{\mu}$ as expected. Therefore all of our integrability conditions are satisfied. When we evaluate the spin coefficients, their relevant directional derivatives and the curvature scalars, then substitute them in eq. (4.43) we get

$$\begin{aligned}
\tilde{\nabla}_{\mathbb{T}}\mathcal{J} &= \frac{-2}{r^2}+\frac{8M(u)}{r^3}, && (4.81)\\
\tilde{\nabla}_{\mathbb{S}}\mathcal{K} &= 0, && (4.82)\\
\mathcal{J}^2 &= \frac{2}{r^2}-\frac{4M(u)}{r^3}, && (4.83)\\
\mathcal{K}^2 &= 0, && (4.84)\\
\mathcal{R}_{\mathcal{W}} &= \frac{4M(u)}{r^3}. && (4.85)
\end{aligned}$$

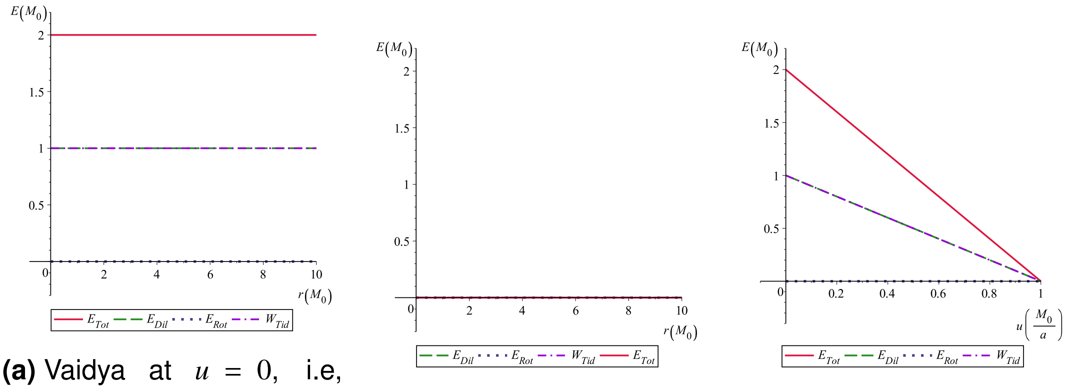
Here we immediately notice that the terms that have been associated with the rotational degrees of freedom, i.e., $\tilde{\nabla}_{\mathbb{S}}\mathcal{K}$ and $\mathcal{K}^2$, are zero. This is expected since Vaidya is a spherically symmetric spacetime.

In order to calculate our quasilocal charges we need to first find the so-called reference curvature k_0. This requires the isometric embedding of the u = constant, r = constant surface to the $\mathcal{M}^4$ which is considered in the spherical coordinates $\{\bar{r},\bar{\theta},\bar{\phi}\}$. For Vaidya, by setting $\{\bar{r}=r,\bar{\theta}=\theta,\bar{\phi}=\phi\}$ we see that the metric induced on $\mathbb{S}$ is trivially isometric to that of the 2-surface embedded in $\mathcal{M}^4$. Then k_0 is given by the scalar curvature of a 2-sphere, i.e., $k_0 = 2/\bar{r} = 2/r$. From eqs. (4.69), (4.73), (4.75) and (4.76) we then have

$$\begin{aligned}
E_{\text{Tot}} &= \frac{-1}{16\pi}\int_{\mathbb{S}} d\mathbb{S}\frac{-\left[2\left(\frac{-2}{r^2}+\frac{8M(u)}{r^3}\right)+\frac{4}{r^2}\right]}{\frac{2}{r}} = 2M(u), && (4.86)\\
E_{\text{Dil}} &= \frac{-1}{16\pi}\int_{\mathbb{S}} d\mathbb{S}\frac{\left[2\left(\frac{2}{r^2}-\frac{4M(u)}{r^3}\right)-\frac{4}{r^2}\right]}{\frac{2}{r}} = M(u), && (4.87)\\
W_{\text{Tid}} &= \frac{-1}{16\pi}\int_{\mathbb{S}} d\mathbb{S}\frac{\left[-2\left(\frac{4M(u)}{r^3}\right)\right]}{\frac{2}{r}} = M(u), && (4.88)\\
E_{\text{Rot}} &= 0. && (4.89)
\end{aligned}$$

Note that we chose a null tetrad in order to satisfy our gauge conditions which turned out to be shear-free. Therefore $H = 2\mathcal{J}^2$ holds in this case and thus $E_{\text{Dil}} = E_{\text{K1}}$. Also, the spacetime Ricci scalar, 24Λ, vanishes. Therefore $\mathcal{R}_{\mathcal{W}} = -2\left(\Psi_2+\overline{\Psi}_2\right) = -4\Psi_2$ and the $\mathcal{R}_{\mathcal{W}}$ term is solely determined by the Coulomb-like gravitational potential.

To visualize a simple evolution, consider mass function $M(u) = M_0 - a u$, where a is a positive constant. These kind of linear mass functions have been used to investigate



(a) Vaidya at $u = 0$, i.e, Schwarzschild geometry. **(b)** Vaidya at $u = 1$, i.e, Minkowski geometry. **(c)** Time evolution of quasilocal charges.

Figure 4.1: Our quasilocal charges give $E_{\mathrm{K1}} = E_{\mathrm{Dil}} = W_{\mathrm{Tid}}$ and $E_{\mathrm{Rot}} = 0$ for each u value. Charges are given in units of M_0 and the time parameter is in units of $M_0/(ac)$ where the speed of light, c, is 1 throughout the thesis.

the black hole evaporation previously in literature (cf. [138], [139], [140]). With this choice of mass function, at $u = 0$ we have the case of a Schwarzschild black hole (See Fig. 4.1a.) which, given enough time, eventually evaporates so that the spacetime becomes Minkowski (See Fig. 4.1b.). The quasilocal charges fall off linearly with the time parameter u (See Fig 4.1c.).

Now let us consider the $\tilde{\nabla}_{\hat{0}} E_{\mathrm{Dil}} = \tilde{\nabla}_{\hat{0}} (E_{\mathrm{K1}}) = E^{\mu}{}_{\hat{0}} \partial_\mu (E_{\mathrm{K1}})$. Following relation (4.18) and with the choices we have made here for **l** and **n**,

$$\tilde{\nabla}_{\hat{0}} E_{\mathrm{Dil}} = \frac{1}{\sqrt{2}} \left[\partial_u + \left(\frac{1}{2} + \frac{M(u)}{r} \right) \partial_r \right] M(u) = \frac{1}{\sqrt{2}} \frac{\partial M(u)}{\partial u}. \tag{4.90}$$

According to the Einstein field equations, $-\frac{2}{r^2} \frac{\partial M(u)}{\partial u} = 8\pi \tilde{\rho}$, where $\tilde{\rho}$ is the energy density of the null dust. This shows that the dilatational energy of the system which could potentially be lost by work, heat or other forms is lost purely due to radiation, for the case of the Vaidya spacetime.

4.5.2 The C-metric

For our second application we want to consider a nonspherically symmetric spacetime. The C-metric is not spherically symmetric and it has many interpretations depending on its coordinate representation. We will consider the coordinate representation which was introduced by Hong and Teo [141],

$$ds^2 = \frac{1}{A^2(x+y)^2}\left(-F(y)d\tau^2 + \frac{dy^2}{F(y)} + \frac{dx^2}{G(x)} + G(x)d\phi^2\right), \tag{4.91}$$

where $G(x) := \left(1-x^2\right)(1+2AMx)$ and $F(y) := -\left(1-y^2\right)(1-2AMy)$. Griffiths *et al.* [142] transformed this cylindrical form of the metric into spherical coordinates by applying the coordinate transformation $\{\tau = At, x = \cos\theta, y = 1/(Ar)\}$ and gave physical interpretations to the C-metric. The transformed metric is written as [142]

$$ds^2 = \frac{1}{(1+Ar\cos\theta)^2}\left(-Q(r)dt^2 + \frac{dr^2}{Q(r)} + \frac{r^2 d\theta^2}{P(\theta)} + P(\theta)r^2\sin^2\theta d\phi^2\right), \tag{4.92}$$

where $Q(r) := \left(1-\frac{2M}{r}\right)\left(1-A^2r^2\right)$ and $P(\theta) := 1+2AM\cos\theta$ with A and M being constants. Note that at $r = 2M$ and at $r = 1/A$ the metric has coordinate singularities and one needs to satisfy the $A^2M^2 < 1/27$ condition in order to preserve the metric signature. Furthermore, eq. (4.92) reduces to the metric of the Schwarzschild black hole in standard curvature coordinates when one sets $A = 0$. Because of this, following Griffiths *et al.* [142], we will interpret the C-metric as the metric of an accelerated black hole. At this point we note that the C-metric is sometimes interpreted as a metric representing two causally disconnected black holes that are joint by a strut and accelerating away from each other [143, 144, 145]. However, this interpretation is valid only when the metric is extended across each horizon, i.e., $r = 2M$ and $r = 1/A$ [142]. For the application of our quasilocal construction we will not consider such an extension of the metric, and the resulting quasilocal charges will correspond to the charges of a single accelerated black hole.

Let us consider the following null tetrad that is generated by the double dyad of the quasilocal observers,

$$\begin{aligned} l^\mu &= \frac{1}{\sqrt{2}}\left[\frac{\Delta}{Q(r)}\right]^{1/2}\partial_t - \frac{1}{\sqrt{2}}\left[\Delta\, Q(r)\right]^{1/2}\partial_r, \\ n^\mu &= \frac{1}{\sqrt{2}}\left[\frac{\Delta}{Q(r)}\right]^{1/2}\partial_t + \frac{1}{\sqrt{2}}\left[\Delta\, Q(r)\right]^{1/2}\partial_r, \\ m^\mu &= \frac{1}{\sqrt{2}}\left[\frac{\Delta\, P(\theta)}{r^2}\right]^{1/2}\partial_\theta + \frac{i}{\sqrt{2}\sin\theta}\left[\frac{\Delta}{r^2 P(\theta)}\right]^{1/2}\partial_\phi, \end{aligned}$$

where $\Delta := (1+Ar\cos\theta)^2$. For such a null tetrad, our integrability conditions $\{\pi+\overline{\tau} = 0, \rho = \overline{\rho}, \mu = \overline{\mu}\}$ hold. The only vanishing spin coefficients are κ, ν, λ and σ, meaning that our null congruences residing on the timelike 2-surface $\mathbb{T}$ are composed of geodesics which are shear-free. As noted earlier this last property is not a necessary condition in our formalism. With the remaining non-vanishing spin coefficients and the variables of the contracted Raychaudhuri equation given in (4.50)-(4.54) we get

$$\begin{aligned}
\tilde{\nabla}_{\mathbb{T}}\mathcal{J} &= \frac{1}{r^3}\left[P(\theta)\left(6r-2A^2r^3\right)-4A\cos\theta r^2+8(M-r)\right], && (4.93)\\
\tilde{\nabla}_{\mathbb{S}}\mathcal{K} &= \frac{2A}{r}\left[2AM\cos^2\theta(2A\cos\theta r+3)+\cos\theta(A\cos\theta r+2)+A(r-2M)\right], && (4.94)\\
\mathcal{J}^2 &= \frac{2Q(r)}{r^2}, && (4.95)\\
\mathcal{K}^2 &= 2A^2P(\theta)\sin^2\theta, && (4.96)\\
\mathcal{R}_{\mathcal{W}} &= 4M\left(\frac{1}{r}+A\cos\theta\right)^3. && (4.97)
\end{aligned}$$

In order to calculate the quasilocal charges we must first calculate the reference energy density, k_0. We isometrically embed $\mathbb{S}$ into $\mathcal{M}^4$, by setting

$$\begin{aligned}
\frac{r^2d\theta^2}{\Delta P(\theta)} &= \bar{r}^2d\bar{\theta}^2, && (4.98)\\
\frac{P(\theta)r^2\sin^2\theta d\phi^2}{\Delta} &= \bar{r}^2\sin^2\bar{\theta}d\bar{\phi}^2. && (4.99)
\end{aligned}$$

and demand that the observers measure the same solid angle in both coordinate systems. This is satisfied by choosing $\bar{r} = r\Delta^{-1/2}$ and then $k_0 = 2/\bar{r}$. Here we should note that for a generic C-metric the angular coordinates are defined within $\{0 < \theta < \pi, -C\pi < \phi < C\pi\}$ where C is the remaining parameter, other than A and M, that parametrizes the spacetime. It is closely related to the 'deficit/excess angle' that tells us how much $\mathbb{S}$ deviates from the spherical symmetry. For example, repeating Griffiths *et al.*'s discussion,

$$\frac{\text{circumference}}{\text{radius}} = \begin{cases} \lim_{\theta\to 0}\frac{2\pi CP(\theta)\sin\theta}{\theta} = 2\pi C(1+2AM) \\ \\ \lim_{\theta\to\pi}\frac{2\pi CP(\theta)\sin\theta}{\pi-\theta} = 2\pi C(1-2AM) \end{cases} \qquad (4.100)$$

shows us that setting $C = 1$, as we choose to do here, will introduce excess and deficit angles on the spacelike surface $\mathbb{S}$ due to the conical singularities that are introduced. This, and our choices for coordinate functions of $\mathcal{M}^4$ will guarantee that the solid angle is the same for the quasilocal observers of the physical and the reference spacetime.

We obtain the quasilocal charges by substituting the quasilocal charge densities, in eqs. (4.93)-(4.97), into the definitions (4.60)-(4.63) and numerically integrating them. The results are presented in Fig. 4.2 for a specific choice of $A = 1/(\sqrt{28}M)$ to perform the numerical integration.

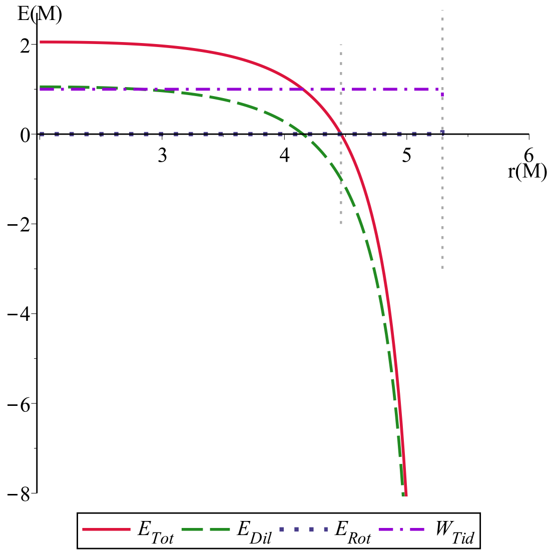


Figure 4.2: Quasilocal charges of the C-metric which is parametrized with $A = \frac{1}{\sqrt{28}M}$. Those quasilocal charges are meaningful only in the region $2M < r < \sqrt{28}M \approx 5.29M$ due to the coordinate singularities.

From Fig. 4.2 we immediately recognise that $E_{\mathrm{K1}} = E_{\mathrm{Dil}}$ decreases as the size of the system increases. For the case of Schwarzschild, i.e., $A = 0$, we expect this curve to be flat, as in Fig. 4.1a. For lower values of acceleration, E_{Dil} gets flatter as expected. This shows that in order for the black hole to be accelerated more, more energy should be input to the system by an *external* agent. In other words, the potential work that can be done *by* the system is lower. Note that after a certain size of the system, E_{Dil} and E_{Tot} take negative values. It may seem counter-intuitive that quasilocal observers could measure a 'negative energy'. To better understand this result, consider the metric (4.92) and define $g_{tt} = -(Q(r)/\Delta) = -[1+2\Phi(r,\theta)]$ where $\Phi(r,\theta)$ plays the role of the 'gravitational potential'. In Fig. 4.3 we plot $\Phi(r,\theta)$ for observers located at different polar angles. We observe that for none of the observers, except the ones located at $\theta = \pi$, $\Phi(r,\theta)$ is monotonic. Moreover, for observers located at $\theta > 0.75\pi$ the gravitational potential changes sign after a certain radial distance. This shows that the effect of the external agent on the system is repulsive. Then the positive total energy $E_{\mathrm{Dil}} + W_{\mathrm{Tid}}$, which corresponds to a system that has an otherwise attractive nature, cannot overcome the repulsive effect of the external agent which causes the black

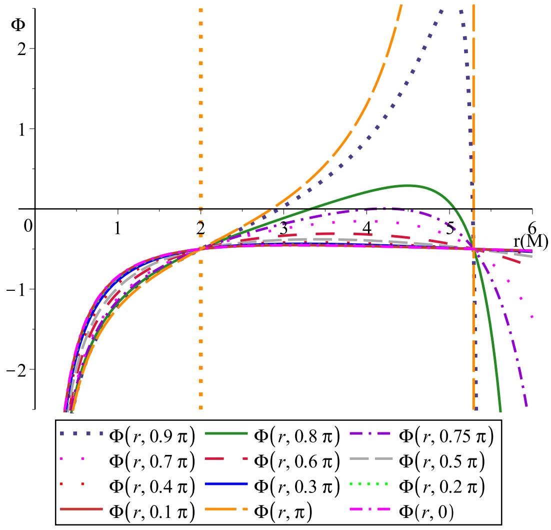
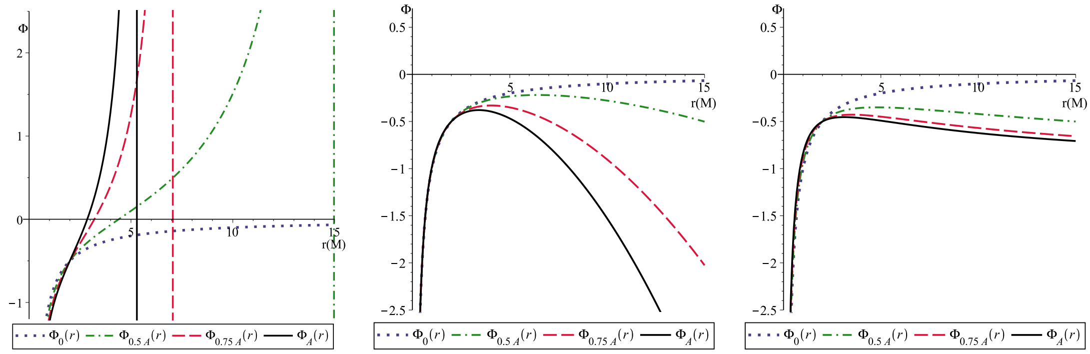


Figure 4.3: Radial behaviour of the gravitational potential of the C-metric, which is parametrized with $A = \frac{1}{\sqrt{28}M}$, plotted for observers located at different polar angles. Those potentials are meaningful only in the region $2M < r < \sqrt{28}M \approx 5.29M$ due to the coordinate singularities.

hole to accelerate. The $E_{\text{Tot}} = 0$ point can be viewed as the minimum energy state of the system, below which it cannot exist without the energy exchange provided by an external agent.

Also recall that the C-metric is interpreted as two black holes which are accelerated *away* from each other. This is a signature of the repulsive behaviour we observe here. Note that here we are investigating one of the most extreme cases for an accelerated black hole, since as for acceleration parameters greater than $1/\left(\sqrt{27}M\right)$ the metric changes signature. Therefore the change in the behaviour of the gravitational potential, and hence a change in the sign of the total energy of the system is not unexpected. We do not observe such behaviour for the Schwarzschild geometry as the gravitational potential is monotonic with constant sign for a static black hole. In order to investigate how the acceleration parameter, A, affects the behaviour of the gravitational potential, see Fig 4.4. We plot $\Phi(r,\theta)$ for observers located at $\theta = \pi$, $\theta = \pi/2$ and $\theta = 0$ respectively in figures 4.4a, 4.4b and 4.4c. For each case, we investigate the effect of the acceleration parameter, A. We observe that only for $A = 0$ case does the gravitational potential not change behaviour. For a more detailed investigation of the behaviour of the gravitational potential of a C-metric, depending on the observer position and on the acceleration parameter, one can see [146].

In order to understand what this means for the acceleration vector of an observer



(a) $\Phi(r,\pi)$ **(b)** $\Phi(r,\pi/2)$ **(c)** $\Phi(r,0)$

Figure 4.4: Acceleration parameter dependence of $\Phi(r,const.)$. For each acceleration parameter A^*, we consider only the region $2M < r < 1/A^*$.

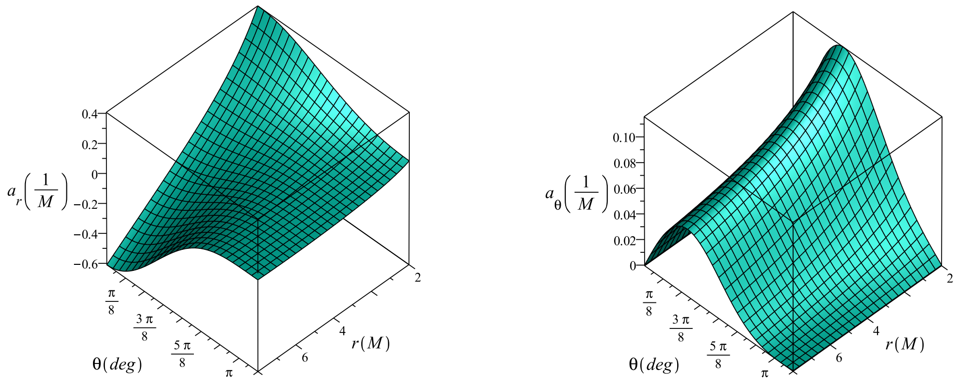


Figure 4.5: Radial and tangential dependence of the components of the acceleration vector. a_r is given on the left and a_θ is given on the right.

of the quasilocal system, let us set the 4-velocity of the observer to be $u^\mu = E^\mu_{\hat{0}} = \frac{1}{\sqrt{2}}(l^\mu + n^\mu)$. Then the acceleration vector is obtained by $a^\mu = D_{E_{\hat{0}}} E^\mu_{\hat{0}} = a_r \partial_r + a_\theta \partial_\theta$ with

$$
\begin{aligned}
a_r &= -\frac{1}{r^2}\left[A^3 r^4 \cos\theta(AM\cos\theta + 1) + A^2 r^2 \cos^2\theta(r - 3M) + A^2 r^2 (r - 2M)\right. \\
&\quad \left. + Ar\cos\theta(r - 4M) - M\right], \qquad (4.101) \\
a_\theta &= \frac{A\sin\theta}{r} P(\theta)\Delta^{1/2}. \qquad (4.102)
\end{aligned}
$$

As it can be seen from Fig. 4.5 the sign of the radial component of the acceleration vector changes depending on the radial and angular position. In Fig. 4.6 we plot the radial dependence of the radial component, a_r, for different observer positions. We observe that for all observers, except the one located at $\theta = \pi$, the direction of the radial acceleration flips. This is due to the change in the behaviour of the gravitational potential and explains why $E_{\rm Dil}$ takes negative values after a critical point.

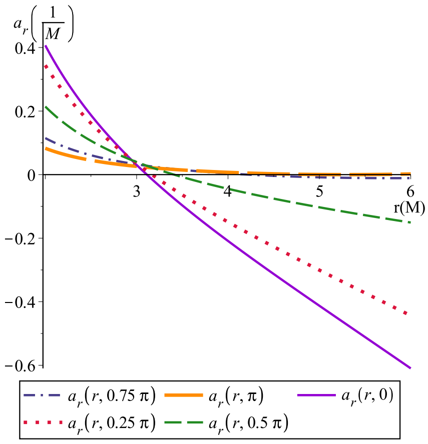


Figure 4.6: Radial behaviour of a_r for observers at different polar angles. We consider the acceleration vector only in the region $2M < r < \sqrt{28}M \approx 5.29M$.

The reason that E_{Dil} and E_{Tot} diverge at $r = \sqrt{28}M$, in Fig. 4.2, results from this point being the second coordinate singularity of our C-metric, as we chose $A = 1/(\sqrt{28}M)$ and the coordinate singularities occur at $\{r = 2M, r = 1/A\}$. This result is expected since after this point, the nature of the spacetime geometry is different.

We also recognise that the system does not possess any energy which can be attributed to rotational degrees of freedom. This is not immediately obvious since the densities (4.94) and (4.96) which appear in definition (4.73) are nonzero. However, what is physical for the quasilocal observers are the quasilocal charges, not the quasilocal densities. Having zero energy associated with the rotational degrees of freedom is expected since the black hole in question is nonrotating.

Finally we observe that the work that has already been done by the tidal fields, W_{Tid}, is positive for all system sizes and takes the same value as in the case of a static black hole. This means that although the individual observers could measure tidal squeezing and tidal stretching depending on their position, the overall effect on the system corresponds to a positive quasilocal charge.

4.5.3 Lanczos-van Stockum dust

For our next application we would like to consider a rotating spacetime. For this, we pick one of the simplest exact solutions of Einstein equations: a rigidly rotating dust cylinder. This solution was first found by Lanczos [147], later rediscovered and matched to a vacuum exterior by van Stockum [148]. Its physicality and mathematical aspects have been investigated intensively in the literature [149, 150, 151, 152, 153, 154, 155, 156]. Also lately, rotating dust metrics have been used to model galaxies in attempts to understand the general relativistic effects on the galaxy rotation curves [157, 158, 159].

The original derivation of van Stockum does not end up with an asymptotically flat spacetime. The energy density of the dust, $\tilde{\rho}$, increases exponentially with increasing cylindrical radial coordinate, x, and it is given by $\tilde{\rho} = \omega^2 e^{\omega^2 x^2}/(2\pi)$. This is not realistic. Later investigations in literature, naturally focus on creating more realistic models which are asymptotically flat. In such cases, components of the line element are given by series solutions [152, 157, 158, 155].

For our application in the current section, we want to focus on finding the quasilocal energy of the spacetime that is associated with the rotational degrees of freedom. We need to find an orthonormal dyad that satisfy the integrability conditions and this already is not an easy task for axially symmetric stationary spacetimes.[7] Therefore we will consider the simplest interior solution given by van Stockum which has a line element

$$ds^2 = -dt^2 + e^{-\omega^2 x^2} dx^2 + e^{-\omega^2 x^2} dz^2 + \left(x^2 - \omega^2 x^4\right) d\phi^2 + 2\omega x^2 dt\, d\phi, \tag{4.103}$$

where ω is a constant that is associated with the angular velocity of the dust at $x = 0$ with respect to 'distant stars'. Other than the singularity at $x = 0$, the spacetime becomes singular at $x = 1/\omega$ for the metric in (4.103). Note that the $g_{\phi\phi}$ component of the metric changes sign when $x > 1/\omega$. This introduces closed timelike curves into the spacetime that are not physical. Therefore we will consider systems within the $0 < x < 1/\omega$ range.

It is possible to transform the metric into spherical coordinates at this point and search for a double dyad which satisfy our gauge conditions (4.31) [8]. Eventually we would like to calculate our quasilocal charges. However, if we apply such a transformation, we lose the information about the actual symmetries of the system. Therefore, let us first

[7] We discuss this in more detail in the next section.

[8] The reason for choosing spherical coordinates is that it simplifies the process of defining a smooth, closed, spacelike 2-surface in order to integrate the quasilocal densities. Without the existence of such a closed surface, quasilocal energies are not defined. This is closely related to the Stokes' Theorem which comes up in the derivation of the non-vanishing boundary Hamiltonian from an action principle of general relativity in a covariant formulation.

consider a null tetrad in cylindrical coordinates which satisfies our gauge conditions, (4.31),

$$
\begin{aligned}
l^{\mu} &= \frac{1}{\sqrt{2}}\left[\partial_t + e^{\omega^2 x^2/2}\partial_x\right], \\
n^{\mu} &= \frac{1}{\sqrt{2}}\left[\partial_t - e^{\omega^2 x^2/2}\partial_x\right], \\
m^{\mu} &= \frac{i}{\sqrt{2}}\left[\omega x\partial_t + \frac{1}{x}\partial_\phi - i e^{\omega^2 x^2/2}\partial_z\right].
\end{aligned} \tag{4.104}
$$

For such a tetrad $\{\pi = 0, \tau = 0\}$ so that the condition $\pi + \overline{\tau} = 0$ is trivially satisfied. Also $\{\mu = \overline{\mu}, \rho = \overline{\rho}\}$ holds. Now we can perform a coordinate transformation by $\{x = r\cos\theta, z = r\sin\theta\}$ to both the metric (4.103) and the tetrad (4.104). We than find

$$ds^2 = -dt^2 + e^{-\omega^2 r^2 \sin^2\theta}\left(dr^2 + r^2 d\theta^2\right) + r^2\sin^2\theta\left(1 - \omega^2 r^2 \sin^2\theta\right)d\phi^2 + 2\omega r^2 \sin^2\theta dt d\phi,$$

and

$$
\begin{aligned}
l^{\mu} &= \frac{1}{\sqrt{2}}\left[\partial_t + \sin\theta e^{\omega^2 r^2 \sin^2\theta/2}\partial_r + \frac{\cos\theta}{r}e^{\omega^2 r^2 \sin^2\theta/2}\partial_\theta\right], \\
n^{\mu} &= \frac{1}{\sqrt{2}}\left[\partial_t - \sin\theta e^{\omega^2 r^2 \sin^2\theta/2}\partial_r - \frac{\cos\theta}{r}e^{\omega^2 r^2 \sin^2\theta/2}\partial_\theta\right], \\
m^{\mu} &= \frac{i}{\sqrt{2}}\left[\omega r\sin\theta\partial_t + \frac{1}{r\sin\theta}\partial_\phi - i\cos\theta e^{\omega^2 r^2 \sin^2\theta/2}\partial_r + i\frac{\sin\theta}{r}e^{\omega^2 r^2 \sin^2\theta/2}\partial_\theta\right].
\end{aligned} \tag{4.105}
$$

For this null tetrad, after calculating the spin coefficients and by following (4.50)-(4.54), we find the following variables that appear in the contracted Raychaudhuri equation,

$$\tilde{\nabla}_{\mathbb{T}}\mathcal{J} = \frac{-e^{\omega^2 r^2 \sin^2\theta/2}\left(r^4\omega^4\sin^4\theta + 1\right)}{r^2\sin^2\theta}, \tag{4.106}$$

$$\tilde{\nabla}_{\mathbb{S}}\mathcal{K} = 0, \tag{4.107}$$

$$\mathcal{J}^2 = \frac{-e^{\omega^2 r^2 \sin^2\theta/2}\left(r^4\omega^4\sin^4\theta + 1\right)}{r^2\sin^2\theta}, \tag{4.108}$$

$$\mathcal{K}^2 = -2\omega^2 e^{\omega^2 r^2 \sin^2\theta/2}, \tag{4.109}$$

$$\mathcal{R}_{\mathcal{W}} = -2\omega^2 e^{\omega^2 r^2 \sin^2\theta/2}. \tag{4.110}$$

In order to determine the reference energy density we isometrically embed $\mathbb{S}$ in $\mathcal{M}^4$

by setting

$$e^{-\omega^2 r^2 \sin^2\theta} r^2 d\theta^2 = \bar{r}^2 d\bar{\theta}^2, \tag{4.111}$$

$$\left(1-\omega^2 r^2 \sin^2\theta\right) r^2 \sin^2\theta d\phi^2 = \bar{r}^2 \sin^2\bar{\theta} d\bar{\phi}^2. \tag{4.112}$$

and once again demand that the observers measure the same solid angle in both spacetimes. Then $\bar{r} = re^{-\omega^2 r^2 \sin^2\theta/4}\left(1-\omega^2 r^2 \sin^2\theta\right)^{1/4}$ and $k_0 = 2/\bar{r}$.

From relations (4.106) and (4.108) we recognize that the corresponding quasilocal charge densities are singular at $\theta = 0$ and $\theta = \pi$. Therefore, for this specific example, we need to take improper integrals to obtain our quasilocal charges, i.e.,

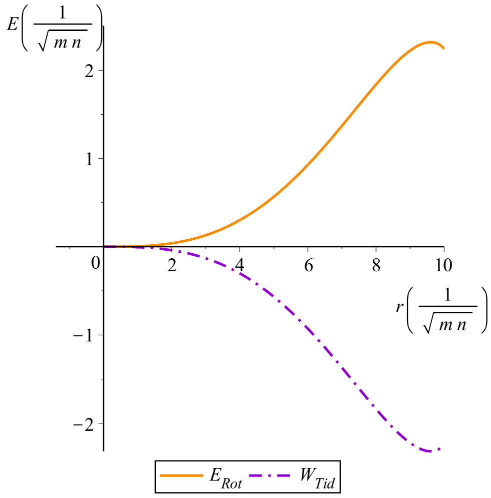
$$E_{\mathrm{Tot}} = \lim_{\epsilon\to 0}\left(-\frac{1}{16\pi}\int_{\phi=-\pi}^{\phi=\pi}\int_{\theta=\epsilon}^{\theta=\pi+\epsilon} d\mathbb{S}\left[\frac{-\left(2\tilde{\nabla}_{\mathbb{T}}\mathcal{J}+k_0^2\right)}{k_0}\right]\right), \tag{4.113}$$

$$E_{\mathrm{Dil}} = \lim_{\epsilon\to 0}\left(-\frac{1}{16\pi}\int_{\phi=-\pi}^{\phi=\pi}\int_{\theta=\epsilon}^{\theta=\pi+\epsilon} d\mathbb{S}\left[\frac{2\mathcal{J}^2-k_0^2}{k_0}\right]\right). \tag{4.114}$$

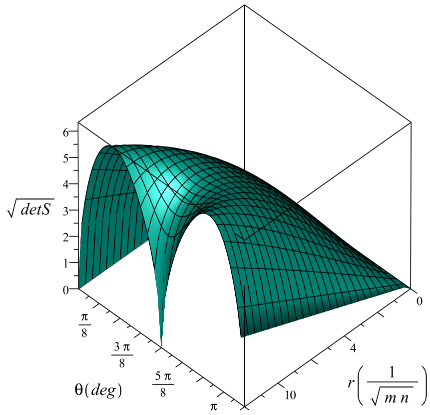
Then we observe that $E_{\mathrm{Tot}} \to -\infty$ and $E_{\mathrm{Dil}} \to -\infty$ for all values of r.

Let us try to understand what this result means. Previously, for asymptotically flat versions of the rotating dust, it has been argued by Bonnor that there has to be an infinitely large negative mass associated with the singularity, $x = 0$, in order to cancel the effect of positive energy associated with the dust [149]. Later in [153] he argued that one can add an infinitely large negative mass layer into the spacetime to observe the same effect. Furthermore Bratek *et al.* [155] discussed the same issue and concluded that singularities of the asymptotically flat rotating dust are associated with the 'additional weird stresses' of the negative active mass.

Here our spacetime is not asymptotically flat. However, we observe a similar behaviour. Note that in our solution the energy density of the dust increases with increasing x. In such a case one would expect the system to get ever closer to a collapsed state as its size increases. Zingg *et al.* [154] and Gurlebeck [156] have argued that such a collapse is in fact expected for a Newtonian dust cylinder. We end up with a similar interpretation which agrees with their arguments. In our work, the infinitely large negative quasilocal dilatational mass-energy must be attributed to the work done by external fields that are required to exist outside our system to prevent the system from collapsing.



(a) Quasilocal charges of the van Stockum dust. E_{Tot} and E_{Dil} diverges to $-\infty$.



(b) Dependence of the 2-surface area element on r and θ coordinates. In spherically symmetric coordinate system singularity is at $\{r = 1/\omega,\ \theta = \pi/2\}$ with $\omega = 1/10$ for our numerical application.

Figure 4.7: Charges are in length units which can be written as a function of individual mass of the dust particles, m, and the total number density, n.

Now let us calculate the quasilocal charges associated with the rotational degrees of freedom and the tidal fields. Once we integrate the quasilocal charge densities defined via equations (4.107), (4.109), (4.110) and k_0 we get the charges plotted in Fig. 4.7a. Note that we picked $\omega = 1/10$ and therefore the radial coordinate $r = \sqrt{x^2+z^2}$ is in the $0 < r < 10$ range in order not to have closed timelike curves. We numerically integrate eq. (4.73) and eq. (4.75) to obtain E_{Rot} and W_{Tid}. From Fig. 4.7a we observe that the W_{Tid} is everywhere negative, corresponding to tidal stretching of the surface on which the quasilocal observers are located. As the size of the system increases, so does the energy density of dust according to $\tilde{\rho} = \omega^2 e^{\omega^2 x^2}/(2\pi)$. This requires greater negative work done by the tidal field. The magnitude of W_{Tid} is exactly equal to the energy associated with the rotational degrees of freedom as shown in Fig. 4.7a. We note that the observers who determine the quasilocal quantities are timelike geodesic observers, i.e., with acceleration $a^\mu = D_{E_{\hat{0}}} E^\mu{}_{\hat{0}} = 0$ and furthermore they are comoving with the dust. In other words, the orbital angular velocity of the observers is zero with respect to the given coordinate system. In such a case one might expect to get zero energy associated with the rotational degrees of freedom of the system. However, for this set of observers, the vorticity of the timelike geodesics is nonzero. Indeed, the vorticity vector and vorticity scalar are given by

$$\begin{aligned} w^\mu &= \frac{1}{2}\eta^\mu{}_{\nu\alpha\beta}g^{\nu\gamma}g^{\alpha\rho}E^\beta{}_{\hat{0}}D_\rho E_{\gamma\hat{0}} = \frac{2\omega\cos\theta e^{2\omega^2r^2\sin^2\theta}}{r^2\sin\theta}\partial_r - \frac{2\omega e^{2\omega^2r^2\sin^2\theta}}{r^3}\partial_\theta \\ w &= \sqrt{w^\mu w_\mu} = \frac{2we^{3\omega^2r^2\sin^2\theta/2}}{r^2\sin\theta}, \end{aligned} \tag{4.115}$$

where $\eta^\mu{}_{\nu\alpha\beta}$ is the Levi-Civita tensor, $g_{\mu\nu}$ is the spacetime metric and we set the observer 4-velocity $u^\mu = E^\mu{}_{\hat{0}} = \partial_t$. This shows that every dust particle swirls around its own axis. Recall that vorticity is a measure of global rotation of a spacetime. Also previously it was shown by Chrobok *et al.* [160] that the rotation of the local matter elements, i.e. spin, can be directly linked to the global rotation of the spacetime, i.e. vorticity. Therefore even though the system we investigate here is defined by the set of observers with zero orbital angular velocity we can still calculate the energy associated with the rotational degrees of freedom of the system.

As the size of the system reaches $1/\omega$, the density of the dust reaches its maximum possible value. Accordingly, one might expect E_{Rot} and W_{Tid} to diverge to $+\infty$ and $-\infty$ respectively at the singularity point $1/\omega$. The peaks we observe in E_{Rot} and W_{Tid} curves of Fig. 4.7a is purely due to the distorted 2-surface area on which we are integrating our quasilocal densities. Fig. 4.7b depicts the area element of $\mathbb{S}$. We observe that at $\{r = 1/\omega, \theta = \pi/2\}$ area element becomes zero. This causes $\mathbb{S}$ to have a distorted shape on the overall. As $\omega \to 0$ and the surface becomes less distorted, E_{Rot} and W_{Tid} diverge to $+\infty$ and $-\infty$ respectively as one expects.

4.6 The challange of stationary, axially symmetric spacetimes

After considering those somewhat unrealistic scenarios one might wonder whether we can apply our formalism to more realistic cases. For example, can we calculate the quasilocal charges of a rotating black hole? The short answer is: yes, we can. However it poses an immense technical challenge.

Recall that we need to satisfy three null tetrad conditions, namely, $\{\rho = \bar{\rho}, \mu = \bar{\mu}, \pi + \bar{\tau} = 0\}$. It is known that in general, the divergence of a null congruence around the vector **l**, can be written as the linear combination of the expansion and the twist of the congruence, i.e., $\rho = \Theta + i\omega$. This means that we need to have nontwisting null congruences for our formalism to hold.

Let us consider the case of the Kerr spacetime [161]. The circular orbits are the mostly studied worldlines of Kerr because the trajectories follow the Killing vector fields and this simplifies the investigations considerably. Note that in this case, the Killing vectors ∂_t and ∂_ϕ have nonzero twist. Moreover, the Kerr metric can be obtained by taking the r coordinate of Schwarzschild to $r + ia\cos\theta$ [162], where a is the dimensionless angular momentum parameter. This automatically means that for a *principal null tetrad* of a static black hole, by transforming the real divergence, $\rho = -1/r$ into a complex divergence $\rho = -1/(r + ia\cos\theta)$, we obtain a rotating black hole.[9] Our problem here is that investigations of a rotating black hole are done mostly using the principal null directions of the spacetime. We should also mention that there are other transverse tetrads such as the quasi-Kinnersely tetrad, which is a powerful tool for exploring Kerr [114]. However, once we focus on such null *geodesics*, that aid in the construction of a principal or transverse tetrad, then we have no hope of finding null congruences with a real divergence.

On the other hand, twist-free – i.e., surface forming – null congruences exist in *all* Lorentzian spacetimes [134]. It is just that we do not require them to be geodesic. Brink *et al.* [164] have given a detailed investigation of axisymmetric spacetimes, focusing on the twist-free Killing vectors of the stationary axially symmetric spacetimes. We note that there are very few studies in literature that investigate such a property. Bilge has found an exact twist-free solution whose principal null directions are not geodesic [165]. It was also shown by Bilge and Gürses that those spacetimes are not asymptotically flat and include generalised Kerr-Schild metrics [166]. Gergely and Perjés later concluded that those solutions are actually homogeneous and isotropic Kasner solutions and they are not physical [167]. Therefore Brink *et al.* conclude that "*Future studies which aim to extract physical information about isolated dynamical, axisymmetric spacetimes will have to focus on general spacetimes, where none of the principal null directions are geodesics, and which do not fall within Bilge's class of metrics.*"

In our case we are looking for *a* null congruence that resides on $\mathbb{T}$, which does not even have to be aligned with the principal null directions. It is not necessarily com-

[9] See [163] for a recent review.

posed of geodesics and it is not required to be composed solely of Killing vectors. All we want from our null tetrad is for it to satisfy the three integrability conditions. To the best of our knowledge, for the case of Kerr, none of the null tetrads introduced in the literature satisfies those conditions.

In order to find such a desired tetrad for the case of Kerr, one might consider the transformations of the quasi-Kinnersely tetrad, for example, by applying two successive Lorentz transformations to the null tetrad. First, apply a Type-II Lorentz transformation around **n** with parameter $A = a + ib$ and then a Type-I Lorentz transformation around **l** with parameter $B = c + id$ where $\{a,\ b,\ c,\ d\}$ are all real. Then for the twice transformed spin coefficients we need to satisfy $\{\rho'' = \overline{\rho}'', \mu'' = \overline{\mu}'', \pi'' + \overline{\tau}'' = 0, \overline{\pi}'' + \tau'' = 0\}$ where $''$ denotes the fact that the spin coefficients are transformed twice. After such a procedure we end up with four complex, highly coupled, non-linear first order differential equations. The unknowns appear in the transformed tetrad condition equations with a polynomial order that goes up to order five. This system of equations cannot be solved by any iterative method that we are aware of.

Therefore, we observe that our formalism should, in principle, be applicable for more realistic generic spacetimes than the ones we have presented here. However, the less symmetry the system possesses, the more mathematically challenging it becomes to find a null tetrad which satisfies our integrability conditions. Arbitrary nontwisting null congruences of twisting spacetimes are the key to resolving this issue.

The discussion we presented in Section 4.2 should now be more clear for the reader. In the case of gravitational wave detection, one's ultimate aim is to extract information about the properties of the astrophysical objects that are the sources of radiation. Those properties, such as mass-energy and angular momentum are at best defined quasilocally in general relativity. Therefore the local tetrads of observers should be chosen in such a manner that the quasilocal properties of the system can be well defined throughout the evolution. In [114], Zhang *et al.* showed that the wave fronts of passing gravitational radiation are aligned with the quasi-Kinnersley tetrad. This means that the observers can measure the gravitational radiation locally. However, since quasi-Kinnersley tetrad does not satisfy the integrability conditions of $\mathbb{S}$ and $\mathbb{T}$, the quasilocal charges corresponding to the quasi-Kinnersley tetrad are not well defined. Therefore we conclude that even though one can measure the gravitational radiation locally, there is not always a guarantee that one can extract the properties of its source consistently.

4.7 Discussion

The energy and energy flux definitions that are made locally, globally or quasilocally, are sometimes compared and contrasted without questioning for which system those definitions are made. Actually, there exist well defined quasilocal energy definitions that can be directly linked to the action principle of general relativity. What is ill-defined is the specification the system that is enclosed by a boundary surface on which the quasilocal charges are to be integrated.

Let us make an analogy with classical thermodynamics and consider two systems: (i) a constant pressure system which is expanding and (ii) a constant volume system which has increasing pressure. If we use a barometer to measure the pressure values obtained within these two systems, the readings will of course be different. However, this is not because the barometer is not working properly, rather it is because the barometer is not sensitive to the defining properties (or symmetries) of the two systems in question. Moreover, even if we find a way to define the system consistently there exist many energies one can associate with a system. Going back to our analogy, let us say we keep track of the pressure value and make sure that we are actually investigating a system with constant pressure. Now we can define the internal energy of that system or define the average kinetic energy of the particles which is not necessarily related to internal energy unless there exist equilibrium. We can also define work done by the system on the surroundings throughout the expansion process etc. In that situation we would not expect all of those energies to give us the same value.

In this chapter, we presented a quasilocal work-energy relation which can be applied to generic spacetimes in order to discuss quasilocal energy exchange. We identified the quasilocal charges associated with the rotational and nonrotational degrees of freedom, in addition to a work term associated with the tidal fields. This construction was possible only after we defined a quasilocal system by constraining the double dyad of the quasilocal observers, which is highly dependent on the symmetries of the spacetime in question.

In Chapter 3, for spherically symmetric systems, we investigated the Raychaudhuri equation of the worldsheet at quasilocal thermodynamic equilibrium, i.e., when the observers are located at the apparent horizon. In the present chapter, we considered more generic spacetimes that are in nonequilibrium with their surroundings. We also relaxed the spherically symmetric condition.

By transforming our equations from Capovilla and Guven's formalism, which is constructed on an orthogonal double dyad, to the Newman-Penrose formalism, which is based on a complex null tetrad, we were able to present the contracted Raychaudhuri equation in terms of the combinations of spin coefficients, their relevant directional derivatives and some of the curvature scalars. We also imposed three null tetrad gauge conditions which result from the integrability conditions of the 2-dimensional timelike surface $\mathbb{T}$ and the 2-dimensional spacelike surface $\mathbb{S}$. This spacelike 2-surface is defined instantaneously and is orthogonal to $\mathbb{T}$ at every point. Our null tetrad gauge conditions are shown to be invariant under Type-III Lorentz transformations which basically corresponds to boosting of the quasilocal observers in the spacelike direction orthogonal to $\mathbb{S}$. Ultimately we realised that, under such gauge conditions, the contracted Raychaudhuri equation is a linear combination of two of the spin field equations of the Newman-Penrose formalism.

Later, we defined certain quasilocal charges via the geometric variables that appear in the contracted Raychaudhuri equation. By choosing the quasilocal energy definitions made by Kijowski [25] as our anchor, we were able to define relevant quasilocal charges for which a physical interpretation would be found. We also showed in Appendix C.2 that all of those quasilocal charges are invariant under Type-III Lorentz transformations. Note that this property is desired for a well defined quasilocal construction, as boosted observers should agree on the fact that they are measuring the charges of the same system.

We applied our formalism to a radiating Vaidya spacetime, a C-metric and an interior solution of the Lanczos-van Stockum dust cylinder. For the case of Vaidya we concluded that the usable energy of the system decreases purely due to radiation. For a C-metric we observed that the greater the acceleration of the black hole is, the more energy should be provided *to* the system *by* an external agent. We concluded that the decreasing trend in the total energy is due to the nonmonotonic, repulsive gravitational potential that can be observed at the exterior region of an extremely accelerated black hole. For the Lanczos-van Stockum dust we considered a nonasymptotically flat case. We obtained an infinitely large negative mass-energy for the usable dilatational energy of the system independent of its size and concluded that this must be attributed to external fields doing work on the system in order to prevent it from collapse. We were also able to obtain the quasilocal energy associated with the rotational degrees of freedom whose magnitude is exactly equal to the one of work done by the tidal fields.

It is true that there exist various open problems and delicate issues related to our construction. To start with, at a given spacetime point one has six tetrad degrees of freedom and we imposed only three null tetrad gauge conditions to our system. That means we have additional freedom to specify a gauge, i.e, to define the quasilocal system. Although there exists no geometrically motivated reason we are aware of in our current approach, one can *choose* additional conditions in order to compare the quasilocal charges of different spacetimes constructed with other well known null tetrad gauges.

Another delicate issue which may or may not be related to our null tetrad gauge freedom is shear. There is no *a priori* reason for us to impose the shear-free condition to the null congruences that reside on $\mathbb{T}$. However, for generic spacetimes, one can find a gauge which satisfies our three gauge conditions more easily once the shear-free condition is imposed. This is primarily because our gauge conditions are trying to locate the set of quasilocal observers in such a configuration that the surface $\mathbb{S}$ is always orthogonal to $\mathbb{T}$. That is natural for radially moving observers of a spherically symmetric system but may hold even if the spacetime is not spherically symmetric. The shear-free condition locates the quasilocal observers as close to as they can get to such a configuration. Note that shear is the fundamental concept of Bondi's mass loss [14] without which gravitational radiation at null infinity cannot be defined. Thus, this automatically raises an issue for quasilocal observers at infinity who would like to measure the Bondi mass loss associated with gravtitational radiation. Investigation of whether or not there exist a gauge which satisfies both the Bondi tetrad and our gauge conditions is left for future work.

Finally, we note that it is technically difficult to satisfy our null tetrad conditions for more realistic, axially symmetric, stationary spacetimes such as Kerr. This difficulty arises from the fact that our approach demands twist-free null congruences on the worldsheet $\mathbb{T}$. However, finding twist-free null congruences for spacetimes whose principal null directions are twisting is a challenge. Although those nongeodesic null congruences that we are after are not physical, their existence will guarantee the fact that the quasilocal system, and the associated quasilocal charges, are all consistently defined.

Recently, a quasilocal energy for the Kerr spacetime has been calculated for stationary observers [168] by using the definition of [169] both for the quasilocal energy and the embedding method for the reference energy. Liu and Tam show that this energy is exactly equal to Brown and York's (BY) quasilocal energy, eq. (2.61). One might wonder how our construction is compared to such an investigation. To start with, the null tetrad constructed from the orthonormal double dyad of the stationary observers in Boyer-Lindquist coordinates has imaginary divergence and hence does not satisfy our null tetrad gauge conditions. Recall that the tetrad conditions we introduced here guarantees the existence of well defined, boost-invariant quasilocal charges. Also note that BY quasilocal energy is not invariant under boosts. Thus, the fact that Liu and Tam end up with the BY quasilocal energy for their quasilocal system defined by stationary observers in Boyer-Lindquist coordinates is no surprise. Therefore, in our view, the calculations of Liu and Tam does not satisfy all the requirements of a genuine quasilocal construction.

5 Conclusion

According to many researchers, including the authors of references [170, 171, 172, 173], the 2+2 picture of general relativity might be more fundamental than the 3+1 approach. Although one might debate this point, the existence of a non vanishing boundary Hamiltonian leads to the necessity of modifying the symplectic structure of the ADM formalism in phase space to obtain a covariant formalism which can directly be linked to the quasilocal charges [25, 174]. Energy definitions, which do not conflict with the equivalence principle, generically involve the extrinsic or/and intrinsic geometry of a closed spacelike 2-surface. However, defining quasilocal charges that are measures of energy and angular momentum for a generic spacetime is often a challenge.

Here we realised that in order to construct well defined matter plus gravitational energy definitions one needs to define the system in question consistently. Given the underlying theory is general relativity, such a procedure is highly geometry dependent. Accordingly, the investigation presented in this thesis emerged from three questions:

- Is there something inherently fundamental about the 2+2 formalism in terms of quasilocal energy definitions?
- If quasilocal energy resides on the intrinsic and/or extrinsic curvature of a closed 2-dimensional spacelike surface, what do the other extrinsic properties of that surface correspond to?
- Can the Raychaudhuri and the other geodesic deviation equations, which have proved their usefulness in terms of physically relevant observables in a 3+1 formalism, be investigated in a 2+2 formalism so that they can be linked to physically meaningful quasilocal charges?

To answer these questions we considered Capovilla and Guven's generalised Raychaudhuri equation given in [32] for a 2-dimensional worldsheet that is embedded in a 4-dimensional spacetime. Also, we recognised Kijowski's Hamiltonian formulation and the quasilocal energies [25] as an anchor to build our own construction.

Since the consistent definition of a system depends highly on the symmetries of the underlying spacetime, we started our investigation with a relatively easy task: spherically symmetric systems at thermodynamic equilibrium. We considered only the radially moving observers so that their timelike dyad resides on a temporal-radial plane, i.e., our 2-dimensional worldsheet, $\mathbb{T}$. Note that by imposing such a condition we made sure that the resulting quasilocal thermodynamic potentials are invariant under the radial boosts. This is just another way of stating that such a set of observers agree on the measurements of the same system.

The contracted Raychaudhuri equation of Capovilla and Guven [32], for the case of the situation depicted above, takes a very simple form. Our grounds for interpreting it thermodynamically are due to the Raychaudhuri equation of $\mathbb{T}$ involving terms that are closely related to the quasilocal observables of a system. In particular, the term $\mathcal{J}^2$ in eq. (3.2) can be linked to the mean extrinsic curvature of a closed spacelike 2-surface $\mathbb{S}$ and hence the boundary Hamiltonian of general relativity. Moreover, the term $\mathcal{R}_{\mathcal{W}}$ can be used to define a genuine relative work density in the $2+2$ formalism unlike its local analogue in the $3+1$ picture, which is only applicable for the neighbouring worldlines of the geodesic deviation.

Eventually we interpreted the contracted Raychaudhuri equation by defining the Helmholtz free energy density, f, via the mean extrinsic curvature of $\mathbb{S}$. One obtains the Helmholtz free energy, $\mathcal{F}$, by taking the integral of f over the closed surface $\mathbb{S}$. We defined our quasilocal thermodynamic equilibrium via the minimization of the Helmholtz free energy, as one would do for a constant temperature system in classical thermodynamics. Then the the contracted Raychaudhuri equation can be written in terms of our Gibbs free energy density and the work density *linearly*.

This relation between the the Gibbs free energy and the work term is the same as in the surface dynamics of the classical theory in which the work density, i.e., the surface tension and the curvature of the surface are linked via the Young-Laplace equation. According to the classical theory, fluids tend to extremize their surfaces to reach equilibrium. At this point the surface tension takes its critical value.

Similarly, our thermodynamic equilibrium condition for a gravitating system corresponds to the minimum mean extrinsic curvature of $\mathbb{S}$, i.e., when $\mathbb{S}$ is located at the generalised apparent horizon of a given spacetime. Considering the fact that the apparent and event horizons of a static black hole coincide, the resemblance between the equations defining black hole mechanics and equilibrium thermodynamics seems to be more than just an analogy. We believe that it is closely related to the boundary Hamiltonian of general relativity, and defines a specific state of an arbitrary system.

Note that at hydrodynamic equilibrium of the classical theory, a system neither expands nor contracts. Moreover, hydrodynamic and thermodynamic equilibrium states of a system coincide when the entropy takes its extremum value. Similarly in our approach, at quasilocal thermodynamic equilibrium, the surface $\mathbb{S}$ neither expands nor contracts when it is perturbed along $\mathbb{T}$. Also, our quasilocal entropy, i.e., the 2-surface area of the apparent horizon, takes its extremum value at this state. Thus, we concluded that our quasilocal thermodynamic equilibrium should coincide with the hydrodynamic equilibrium.

In astrophysics, these two coincident equilibrium states signal the existence of a virial relation in which the temporal average of the kinetic energy of the total system becomes equal to the ensemble average of the kinetic energies of the particles at a given time. Then the internal energy, which is equal to the ensemble average of the kinetic energy of the particles at equilibrium, can be directly linked to the time averaged potential energy of the system. For a similar construction in general relativity we extended Bizon, Malec and Ó Murchadha's mass bound [102] obtained for a static black hole to the generic spherically symmetric systems. We picked Kijowski's E_{K1} as our internal energy, which reduces to the Misner-Sharp-Hernandez energy for spherically symmetric spacetimes. We considered E_{KLY} as the proper/invariant mass of the system following Epp's ideas [26] and ended up with a relation which relates the internal energy of the quasilocal system to the potential energy. This relation has the same form as the ultrarelativistic virial relation in astrophysics.

Later on, since we wanted to investigate more generic systems, we considered systems in nonequilibrium and we relaxed the spherically symmetric condition for the spacetime in question. Then our investigation turned into a search for how to best define the quasilocal energy of a system which can potentially be exchanged with the surroundings. Note that in this generalization we do not mention anything about nonequilibrium thermodynamic potentials or relations. This is because even in the classical theory, nonequilibrium thermodynamics is a less investigated territory. Only for systems that are close to equilibrium one can define thermodynamic relations that are diverging from linearity by a small amount. There is no putative, well defined thermodynamic relation for systems far from equilibrium, in which the thermodynamic potentials are related to each other nonlinearly. Therefore, since we wanted to consider those systems defined far from the apparent horizon we abandoned the thermodynamic approach.

In order to better understand how the worldsheet focusing relates to the null cone observables we transformed the contracted Raychaudhuri equation of $\mathbb{T}$, that is written in Capovilla and Guven's notation, to the notation of the Newman-Penrose (NP) formalism [34]. Similarly to the previous case for an equilibrium state, we defined a system via the domain enclosed by $\mathbb{S}$ which is orthogonal to $\mathbb{T}$ at every point. For this, we imposed the integrability conditions on the tangent vectors of $\mathbb{T}$ and $\mathbb{S}$. When the formalism transformation is applied, these integrability conditions correspond to three null tetrad gauge conditions, namely $\tau+\overline{\pi}=0$, $\rho=\overline{\rho}$ and $\mu=\overline{\mu}$, for a null tetrad constructed from the double dyad of the quasilocal observers. By satisfying these conditions one defines a generic, well defined quasilocal system throughout its evolution.

We identified the spin coefficients that are related to the rotational and nonrotational degrees of freedom of the system and isolated the terms in the contracted Raychaudhuri equation accordingly. This, and our previous discussion about the relative work density via the worldsheet deviation, allowed us to define boost-invariant quasilocal energy-like charges. Then we ended up with a work-energy relation which is applicable for systems that can potentially exchange energy.

In terms of the applications, we considered systems that are at quasilocal thermodynamic equilibrium defined in Schwarzschild, Friedmann-Lemaître-Robertson-Walker and Lemaître-Tolman spacetimes. We found that the worldsheet-constant temperature of the system is same as the temperature of the particle tunnelling through the apparent horizon of the given spacetime that is presented in various studies.

We also considered systems in a radiating Vaidya spacetime, a C-metric and a Lanczos-van Stockum dust in order to investigate quasilocal charges in nonequilibrium. For systems defined in Vaidya spacetime we observed that the mass-energy loss is purely due to radiation. For the C-metric and the Lanczos-van Stockum dust metric cases we investigated the effects of the mass-energy input of an external agent to the system in question without which it would not survive.

In this thesis, we proposed a geometric method in order to define and investigate systems in general relativity. Our main outcome is that *without a well defined quasilocal system, there is no consistent definition of energy*. We hope that the realization of this will encourage other researchers who are working on the quasilocal energy formulations in terms of better *system* definitions like ours or Epp *et al.*'s [123].

We discussed the difficulties and delicate issues encountered in our construction in the previous chapter. This basically relates to the fact that, for generic spacetimes, it is not always so easy to find a null tetrad which satisfies our three tetrad conditions. As the system possesses less symmetry this task gets harder. For future work, our primary goal is to investigate whether or not there is a systematic way of finding such a null tetrad, which satisfies the required conditions for an arbitrary observer set.

Moreover, there are further fundamental questions to ask about our construction, or any other quasilocal formulation given in literature, which is formulated in classical general relativity. For example, consider classical quantum mechanics where it is the physical observables such as energy or angular momentum that are quantized. This is a different focus from that of researchers working on quantum gravity who usually tend to quantize *spacetime itself* in order to quantize gravity. However, as it has been mentioned here and has been shown in the literature many times, energy and angular momentum are already quasilocal in general relativity and the spacetime metric does not have an explicit role in their definition. Are we then on the right track in terms of what we should be quantizing?

In [175], Carlip argues for the advantages of doing quantum gravity in $2+1$ dimensions over $3+1$ dimensions. Quantization in a $2+1$ scenario is easier simply because there are no propagating degrees of freedom in the corresponding quantum gravity theory. The spacetime is either locally flat, de Sitter or anti-de Sitter. Note that his construction is valid for a spacetime manifold with closed topology. On the other hand, there have been certain attempts to quantize gravity for manifolds with non-vanishing boundary. The action and the relevant boundary conditions one needs to pick have been discussed for certain situations [176, 177, 178, 179, 180].

However, in all of these investigations the systems evolving from an initial state to a final state have been parametrised by “time”. It is true that the generator of energy is the *single* time parameter that quantifies *change* in Newton’s theory. In general relativity, on the other hand, energy resides on the extrinsic geometry of $\mathbb{S}$ defined via the *two* degrees of freedom lying on the worldsheet $\mathbb{T}$. Therefore, each state of a given system should be parametrised by these two variables. Then the corresponding transition amplitude of the states may be written as a double integral rather than a single, time, integral.

Also, for such an approach, there would be no *problem of time* related to the parametrization invariance of a given foliation in the $3+1$ picture.[1] We believe the $3+1$ formulation is the closest one can get to Newtonian intuition in a relativistic theory. However, recall that abandoning Newtonian intuition yielded incredible results in physics at the first quarter of the twentieth century. Therefore, as a closing remark we suggest that it might be more fruitful to try extending quantization studies in $2+1$ dimensions to $2+2$ dimensions rather than $3+1$. This is just another way of saying that quasilocal energy research should fundamentally underpin more quantum gravity investigations. In the end, whoever has a better insight in terms of the boundary Hamiltonians in a gravitational theory will probably crack the mysteries behind the quantization of gravity.

[1] See [181] for a review.

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A Newman-Penrose Formalism

The formalism was introduced by Newman and Penrose in 1961 and is usually referred to as the spin field formalism or simply the Newman-Penrose formalism in literature. This formalism has the following advantages [182]:

- The Einstein field equations can be written without the usage of index or summation notation.

- The NP spin field equations are first order and one can get useful sets of linear equations by certain groupings of them.

- In this formalism, all of the equations are complex. This means that the total number of equations are halved.

- The NP spin field equations are all scalar and one can carry out an investigation by focusing on only a few of them depending on the geometry of the problem.

We now introduce the NP formalism by considering a complex null tetrad $\{l_a, n_a, m_a, \overline{m}_a\}$. Here the components of **l** and **n** are real and the ones of **m** and $\overline{\mathbf{m}}$ are complex. When we take the inner products of these null vectors with each other, the only non-vanishing inner products are $\langle \mathbf{l}, \mathbf{n} \rangle = -1$ and $\langle \mathbf{m}, \overline{\mathbf{m}} \rangle = 1$. Note that we are using the $\{-,+,+,+\}$ signature for the spacetime metric through out the thesis. Therefore our spin coefficients and the curvature scalars will have an extra negative sign when compared to Newman-Penrose's original notation [34]. However, signature choice does not affect the NP spin field equations or any of the Lorentz transformation relations of spin coefficients.

A.1 Spin coefficients and curvature scalars

The spin coefficients are defined via the changes of null vectors when they are propagated along each other with the relevant projections, i.e.,

$$\kappa = -\langle D_{\mathbf{l}}\mathbf{l}, \mathbf{m}\rangle, \qquad \nu = \langle D_{\mathbf{n}}\mathbf{n}, \overline{\mathbf{m}}\rangle, \tag{A.1}$$

$$\rho = -\langle D_{\overline{\mathbf{m}}}\mathbf{l}, \mathbf{m}\rangle, \qquad \mu = \langle D_{\mathbf{m}}\mathbf{n}, \overline{\mathbf{m}}\rangle, \tag{A.2}$$

$$\sigma = -\langle D_{\mathbf{m}}\mathbf{l}, \mathbf{m}\rangle, \qquad \lambda = \langle D_{\overline{\mathbf{m}}}\mathbf{n}, \overline{\mathbf{m}}\rangle, \tag{A.3}$$

$$\tau = -\langle D_{\mathbf{n}}\mathbf{l}, \mathbf{m}\rangle, \qquad \pi = \langle D_{\mathbf{l}}\mathbf{n}, \overline{\mathbf{m}}\rangle, \tag{A.4}$$

$$\varepsilon = \frac{1}{2}\left[-\langle D_{\mathbf{l}}\mathbf{l}, \mathbf{n}\rangle + \langle D_{\mathbf{l}}\mathbf{m}, \overline{\mathbf{m}}\rangle\right], \tag{A.5}$$

$$\gamma = \frac{1}{2}\left[\langle D_{\mathbf{n}}\mathbf{n}, \mathbf{l}\rangle - \langle D_{\mathbf{n}}\overline{\mathbf{m}}, \mathbf{m}\rangle\right], \tag{A.6}$$

$$\beta = \frac{1}{2}\left[-\langle D_{\mathbf{m}}\mathbf{l}, \mathbf{n}\rangle + \langle D_{\mathbf{m}}\mathbf{m}, \overline{\mathbf{m}}\rangle\right], \tag{A.7}$$

$$\alpha = \frac{1}{2}\left[\langle D_{\overline{\mathbf{m}}}\mathbf{n}, \mathbf{l}\rangle - \langle D_{\overline{\mathbf{m}}}\overline{\mathbf{m}}, \mathbf{m}\rangle\right], \tag{A.8}$$

and the propagation equations follow as

$$D_{\mathbf{l}}\mathbf{l} = (\varepsilon+\bar{\varepsilon})\mathbf{l} - \bar{\kappa}\mathbf{m} - \kappa\overline{\mathbf{m}}, \tag{A.9}$$

$$D_{\mathbf{n}}\mathbf{l} = (\gamma+\bar{\gamma})\mathbf{l} - \bar{\tau}\mathbf{m} - \tau\overline{\mathbf{m}}, \tag{A.10}$$

$$D_{\mathbf{m}}\mathbf{l} = (\bar{\alpha}+\beta)\mathbf{l} - \bar{\rho}\mathbf{m} - \sigma\overline{\mathbf{m}}, \tag{A.11}$$

$$D_{\mathbf{l}}\mathbf{n} = -(\varepsilon+\bar{\varepsilon})\mathbf{n} + \pi\mathbf{m} + \bar{\pi}\overline{\mathbf{m}}, \tag{A.12}$$

$$D_{\mathbf{n}}\mathbf{n} = -(\gamma+\bar{\gamma})\mathbf{n} + \nu\mathbf{m} + \bar{\nu}\overline{\mathbf{m}}, \tag{A.13}$$

$$D_{\mathbf{m}}\mathbf{n} = -(\bar{\alpha}+\beta)\mathbf{n} + \mu\mathbf{m} + \bar{\lambda}\overline{\mathbf{m}}, \tag{A.14}$$

$$D_{\mathbf{l}}\mathbf{m} = \bar{\pi}\mathbf{l} - \kappa\mathbf{n} + (\varepsilon-\bar{\varepsilon})\mathbf{m}, \tag{A.15}$$

$$D_{\mathbf{n}}\mathbf{m} = \bar{\nu}\mathbf{l} - \tau\mathbf{n} + (\gamma-\bar{\gamma})\mathbf{m}, \tag{A.16}$$

$$D_{\mathbf{m}}\mathbf{m} = \bar{\lambda}\mathbf{l} - \sigma\mathbf{n} + (-\bar{\alpha}+\beta)\mathbf{m}, \tag{A.17}$$

$$D_{\mathbf{m}}\overline{\mathbf{m}} = \mu\mathbf{l} - \bar{\rho}\mathbf{n} + (\bar{\alpha}-\beta)\overline{\mathbf{m}}. \tag{A.18}$$

Newman and Penrose introduce two sets of curvature scalars, Weyl scalars and Ricci scalars, which carry the same information as in the Riemann curvature tensor. The

Ricci scalars are defined as

$$
\begin{aligned}
\Phi_{00} &:= \frac{1}{2}R_{\mu\nu}l^\mu l^\nu, \qquad \Phi_{11} := \frac{1}{4}R_{\mu\nu}(l^\mu n^\nu + m^\mu \bar{m}^\nu), \qquad \Phi_{22} := \frac{1}{2}R_{\mu\nu}n^\mu n^\nu, \qquad \Lambda := \frac{R}{24}. \\
\Phi_{01} &:= \frac{1}{2}R_{\mu\nu}l^a m^\nu, \qquad \Phi_{10} := \frac{1}{2}R_{\mu\nu}l^\mu \bar{m}^\nu = \overline{\Phi_{01}}, \qquad \Phi_{02} := \frac{1}{2}R_{\mu\nu}m^\mu m^\nu, \qquad (A.19) \\
\Phi_{20} &:= \frac{1}{2}R_{\mu\nu}\bar{m}^\mu \bar{m}^\nu = \overline{\Phi_{02}} \qquad \Phi_{12} := \frac{1}{2}R_{\mu\nu}m^\mu n^\nu, \qquad \Phi_{21} := \frac{1}{2}R_{\mu\nu}\bar{m}^\mu n^\nu = \overline{\Phi_{12}}.
\end{aligned}
$$

in which $R_{\mu\nu}$ is the Ricci tensor of the spacetime, $\Phi_{00}, \Phi_{11}, \Phi_{22}, \Lambda$ are real scalars and $\Phi_{10}, \Phi_{20}, \Phi_{21}$ are complex scalars. The Weyl scalars are defined as

$$\psi_0 = C_{\mu\nu\alpha\beta}l^\mu m^\nu l^\alpha m^\beta, \tag{A.20}$$

$$\psi_1 = C_{\mu\nu\alpha\beta}l^\mu n^\nu l^\alpha m^\beta, \tag{A.21}$$

$$\psi_2 = C_{\mu\nu\alpha\beta}l^\mu m^\nu \overline{m}^\alpha n^\beta, \tag{A.22}$$

$$\psi_3 = C_{\mu\nu\alpha\beta}l^\mu n^\nu \overline{m}^\alpha n^\beta, \tag{A.23}$$

$$\psi_4 = C_{\mu\nu\alpha\beta}n^\mu \overline{m}^\nu n^\alpha \overline{m}^\beta. \tag{A.24}$$

with $C_{\mu\nu\alpha\beta}$ being the Weyl tensor.

A.2 Spin field equations

We now have enough information to write down the Einstein equations as a set of linear, first order, complex, scalar equations. These spin field equations are found to be

$$D_{\mathbf{l}}\rho - D_{\overline{\mathbf{m}}}\kappa = (\rho^2 + \sigma\bar{\sigma}) + (\varepsilon + \bar{\varepsilon})\rho - \bar{\kappa}\tau - \kappa(3\alpha + \bar{\beta} - \pi) + \Phi_{00}, \tag{A.25}$$

$$D_{\mathbf{l}}\sigma - D_{\mathbf{m}}\kappa = (\rho + \bar{\rho})\sigma + (3\varepsilon - \bar{\varepsilon})\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa + \Psi_0, \tag{A.26}$$

$$D_{\mathbf{l}}\tau - D_{\mathbf{n}}\kappa = (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma + (\varepsilon - \bar{\varepsilon})\tau - (3\gamma + \bar{\gamma})\kappa + \Psi_1 + \Phi_{01}, \tag{A.27}$$

$$D_{\mathbf{l}}\alpha - D_{\overline{\mathbf{m}}}\varepsilon = (\rho + \bar{\varepsilon} - 2\varepsilon)\alpha + \beta\bar{\sigma} - \bar{\beta}\varepsilon - \kappa\lambda - \bar{\kappa}\gamma + (\varepsilon + \rho)\pi + \Phi_{10}, \tag{A.28}$$

$$D_{\mathbf{l}}\beta - D_{\mathbf{m}}\varepsilon = (\alpha+\pi)\sigma + (\bar{\rho}-\bar{\varepsilon})\beta - (\mu+\gamma)\kappa - (\bar{\alpha}-\bar{\pi})\varepsilon + \Psi_1\,, \tag{A.29}$$

$$D_{\mathbf{l}}\gamma - D_{\mathbf{n}}\varepsilon = (\tau+\bar{\pi})\alpha + (\bar{\tau}+\pi)\beta - (\varepsilon+\bar{\varepsilon})\gamma - (\gamma+\bar{\gamma})\varepsilon + \tau\pi - \nu\kappa + \Psi_2 + \Phi_{11} - \Lambda\,, \tag{A.30}$$

$$D_{\mathbf{l}}\lambda - \bar{\delta}\pi = (\rho\lambda + \bar{\sigma}\mu) + \pi^2 + (\alpha-\bar{\beta})\pi - \nu\bar{\kappa} - (3\varepsilon-\bar{\varepsilon})\lambda + \Phi_{20}\,, \tag{A.31}$$

$$D_{\mathbf{l}}\mu - D_{\mathbf{m}}\pi = (\bar{\rho}\mu + \sigma\lambda) + \pi\bar{\pi} - (\varepsilon+\bar{\varepsilon})\mu - (\bar{\alpha}-\beta)\pi - \nu\kappa + \Psi_2 + 2\Lambda\,, \tag{A.32}$$

$$D_{\mathbf{l}}\nu - D_{\mathbf{n}}\pi = (\pi+\bar{\tau})\mu + (\bar{\pi}+\tau)\lambda + (\gamma-\bar{\gamma})\pi - (3\varepsilon+\bar{\varepsilon})\nu + \Psi_3 + \Phi_{21}\,, \tag{A.33}$$

$$D_{\mathbf{n}}\lambda - D_{\overline{\mathbf{m}}}\nu = -(\mu+\bar{\mu})\lambda - (3\gamma-\bar{\gamma})\lambda + (3\alpha+\bar{\beta}+\pi-\bar{\tau})\nu - \Psi_4\,, \tag{A.34}$$

$$D_{\mathbf{m}}\rho - D_{\overline{\mathbf{m}}}\sigma = \rho(\bar{\alpha}+\beta) - \sigma(3\alpha-\bar{\beta}) + (\rho-\bar{\rho})\tau + (\mu-\bar{\mu})\kappa - \Psi_1 + \Phi_{01}\,, \tag{A.35}$$

$$D_{\mathbf{m}}\alpha - D_{\overline{\mathbf{m}}}\beta = (\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho-\bar{\rho}) + \varepsilon(\mu-\bar{\mu}) - \Psi_2 + \Phi_{11} + \Lambda\,, \tag{A.36}$$

$$D_{\mathbf{m}}\lambda - D_{\overline{\mathbf{m}}}\mu = (\rho-\bar{\rho})\nu + (\mu-\bar{\mu})\pi + (\alpha+\bar{\beta})\mu + (\bar{\alpha}-3\beta)\lambda - \Psi_3 + \Phi_{21}\,, \tag{A.37}$$

$$D_{\mathbf{m}}\nu - D_{\mathbf{n}}\mu = (\mu^2 + \lambda\bar{\lambda}) + (\gamma+\bar{\gamma})\mu - \bar{\nu}\pi + (\tau-3\beta-\bar{\alpha})\nu + \Phi_{22}\,, \tag{A.38}$$

$$D_{\mathbf{m}}\gamma - D_{\mathbf{n}}\beta = (\tau-\bar{\alpha}-\beta)\gamma + \mu\tau - \sigma\nu - \varepsilon\bar{\nu} - (\gamma-\bar{\gamma}-\mu)\beta + \alpha\bar{\lambda} + \Phi_{12}\,, \tag{A.39}$$

$$D_{\mathbf{m}}\tau - D_{\mathbf{n}}\sigma = (\mu\sigma + \bar{\lambda}\rho) + (\tau+\beta-\bar{\alpha})\tau - (3\gamma-\bar{\gamma})\sigma - \kappa\bar{\nu} + \Phi_{02}\,, \tag{A.40}$$

$$D_{\mathbf{n}}\rho - D_{\overline{\mathbf{m}}}\tau = -(\rho\bar{\mu} + \sigma\lambda) + (\bar{\beta}-\alpha-\bar{\tau})\tau + (\gamma+\bar{\gamma})\rho + \nu\kappa - \Psi_2 - 2\Lambda\,, \tag{A.41}$$

$$D_{\mathbf{n}}\alpha - D_{\overline{\mathbf{m}}}\gamma = (\rho+\varepsilon)\nu - (\tau+\beta)\lambda + (\bar{\gamma}-\bar{\mu})\alpha + (\bar{\beta}-\bar{\tau})\gamma - \Psi_3. \tag{A.42}$$

Also note that the following commutation relations, $[\mathbf{X},\mathbf{Y}] = D_{\mathbf{X}}\mathbf{Y} - D_{\mathbf{Y}}\mathbf{X}$, for the null vectors, $\{l_a, n_a, m_a, \overline{m}_a\}$, that act on scalars will be relevant for our investigation,

$$[\mathbf{l},\mathbf{n}] = -(\gamma+\overline{\gamma})\,\mathbf{l} - (\varepsilon+\overline{\varepsilon})\,\mathbf{n} + (\pi+\overline{\tau})\,\mathbf{m} + (\overline{\pi}+\tau)\,\overline{\mathbf{m}}, \tag{A.43}$$

$$[\mathbf{l},\mathbf{m}] = (\overline{\pi}-\overline{\alpha}-\beta)\,\mathbf{l} - \kappa\mathbf{n} + (\varepsilon-\overline{\varepsilon}+\overline{\rho})\,\mathbf{m} + \sigma\overline{\mathbf{m}}, \tag{A.44}$$

$$[\mathbf{n},\mathbf{m}] = \overline{\nu}\mathbf{l} + (\overline{\alpha}+\beta-\tau)\,\mathbf{n} + (\gamma-\overline{\gamma}-\mu)\,\mathbf{m} - \overline{\lambda}\overline{\mathbf{m}}, \tag{A.45}$$

$$[\mathbf{m},\overline{\mathbf{m}}] = (\mu-\overline{\mu})\,\mathbf{l} + (\rho-\overline{\rho})\,\mathbf{n} + \left(\overline{\beta}-\alpha\right)\mathbf{m} + (\overline{\alpha}-\beta)\,\overline{\mathbf{m}}. \tag{A.46}$$

In Chapter 4, Lorentz transformation of the null tetrad are of interest. Therefore we will remind the reader of the three types of Lorentz transformations of the NP formalism now.

A.3 Type-I Lorentz transformations

In a Type-I Lorentz transformation, the null tetrad is rotated around the vector $\mathbf{l}$ so that the tetrad vectors transform as

$$\mathbf{l} \rightarrow \mathbf{l}, \tag{A.47}$$

$$\mathbf{n} \rightarrow c\bar{c}\mathbf{l}+\mathbf{n}+\bar{c}\mathbf{m}+c\overline{\mathbf{m}}, \tag{A.48}$$

$$\mathbf{m} \rightarrow c\mathbf{l}+\mathbf{m}, \tag{A.49}$$

$$\overline{\mathbf{m}} \rightarrow \bar{c}\mathbf{l}+\overline{\mathbf{m}}. \tag{A.50}$$

Here c is a constant which is complex. Accordingly the spin coefficients transform as

$$\begin{aligned} \nu \rightarrow\ & \nu+c\bar{c}\pi+c\lambda+\bar{c}\mu+\bar{c}^2\tau+\bar{c}^3c\kappa+\bar{c}^2c\rho+\bar{c}^3\sigma+2\bar{c}\gamma+2\bar{c}^2c\varepsilon+2c\bar{c}\alpha+2\bar{c}^2\beta \\ +\ & D_{\mathbf{n}}\bar{c}+c\bar{c}D_{\mathbf{l}}\bar{c}+cD_{\overline{\mathbf{m}}}\bar{c}+\bar{c}D_{\mathbf{m}}\bar{c}, \end{aligned} \tag{A.51}$$

$$\tau \rightarrow \tau+\bar{c}\sigma+c\rho+c\bar{c}\kappa, \tag{A.52}$$

$$\gamma \rightarrow \gamma+c\bar{c}\varepsilon+c\alpha+\bar{c}\beta+\bar{c}\tau+\bar{c}^2c\kappa+c\bar{c}\rho+\bar{c}^2\sigma, \tag{A.53}$$

$$\mu \rightarrow \mu+c\pi+\bar{c}^2\sigma+\bar{c}^2c\kappa+2\bar{c}\beta+2c\bar{c}\varepsilon+D_{\mathbf{m}}\bar{c}+cD_{\mathbf{l}}\bar{c}, \tag{A.54}$$

$$\sigma \rightarrow \sigma+c\kappa, \tag{A.55}$$

$$\beta \rightarrow \beta+c\varepsilon+\bar{c}\sigma+c\bar{c}\kappa, \tag{A.56}$$

$$\lambda \rightarrow \lambda+\bar{c}\pi+2\bar{c}\alpha+2\bar{c}^2\varepsilon+\bar{c}^2\rho+\bar{c}^3\kappa+D_{\overline{\mathbf{m}}}\bar{c}+\bar{c}D_{\mathbf{l}}\bar{c}, \tag{A.57}$$

$$\rho \rightarrow \rho+\bar{c}\kappa, \tag{A.58}$$

$$\alpha \rightarrow \alpha+\bar{c}\varepsilon+\bar{c}\rho+\bar{c}^2\kappa, \tag{A.59}$$

$$\kappa \rightarrow \kappa, \tag{A.60}$$

$$\varepsilon \rightarrow \varepsilon+\bar{c}\kappa, \tag{A.61}$$

$$\pi \rightarrow \pi+2\bar{c}\varepsilon+\bar{c}^2\kappa+D_{\mathbf{l}}\bar{c}. \tag{A.62}$$

The tranformations of the Ricci scalars are given by

$$\Phi_{00} \rightarrow \Phi_{00}, \tag{A.63}$$
$$\Phi_{01} \rightarrow \Phi_{01} + c\Phi_{00}, \tag{A.64}$$
$$\Phi_{10} \rightarrow \Phi_{10} + \bar{c}\Phi_{00}, \tag{A.65}$$
$$\Phi_{02} \rightarrow \Phi_{02} + 2c\Phi_{01} + c^2\Phi_{00}, \tag{A.66}$$
$$\Phi_{20} \rightarrow \Phi_{20} + 2\bar{c}\Phi_{10} + \bar{c}^2\Phi_{00}, \tag{A.67}$$
$$\Phi_{11} \rightarrow \Phi_{11} + c\bar{c}\Phi_{00} + c\Phi_{10} + \bar{c}\Phi_{01}, \tag{A.68}$$
$$\Phi_{12} \rightarrow \Phi_{12} + 2c\bar{c}\Phi_{01} + c^2\Phi_{10} + \bar{c}\Phi_{02} + 2c\Phi_{11} + c^2\bar{c}\Phi_{00}, \tag{A.69}$$
$$\Phi_{21} \rightarrow \Phi_{21} + 2c\bar{c}\Phi_{10} + \bar{c}^2\Phi_{01} + c\Phi_{20} + 2\bar{c}\Phi_{11} + \bar{c}^2c\Phi_{00}, \tag{A.70}$$
$$\begin{aligned}\Phi_{22} \rightarrow\ & \Phi_{22} + 4c\bar{c}\Phi_{11} + 2c\Phi_{21} + 2\bar{c}\Phi_{12} + \bar{c}^2c^2\Phi_{00} + 2\bar{c}c^2\Phi_{10} \\ +\ & 2\bar{c}^2c\Phi_{01} + \bar{c}^2\Phi_{02} + c^2\Phi_{20},\end{aligned} \tag{A.71}$$

and the transformations of the Weyl scalars are given by

$$\Psi_0 \rightarrow \Psi_0, \tag{A.72}$$
$$\Psi_1 \rightarrow \Psi_1 + \bar{c}\Psi_0, \tag{A.73}$$
$$\Psi_2 \rightarrow \Psi_2 + 2\bar{c}\Psi_1 + \bar{c}^2\Psi_0, \tag{A.74}$$
$$\Psi_3 \rightarrow \Psi_3 + 3\bar{c}\Psi_2 + 3\bar{c}^2\Psi_1 + \bar{c}^3\Psi_0, \tag{A.75}$$
$$\Psi_4 \rightarrow \Psi_4 + 4\bar{c}\Psi_3 + 6\bar{c}^2\Psi_2 + 4\bar{c}^3\Psi_1 + \bar{c}^4\Psi_0. \tag{A.76}$$

A.4 Type-II Lorentz transformations

In Type-II Lorentz transformation, the null tetrad is rotated around the vector **n** and the tetrad vectors transform as

$$\mathbf{l} \rightarrow \mathbf{l} + c\bar{c}\mathbf{n} + \bar{c}\mathbf{m} + c\bar{\mathbf{m}}, \tag{A.77}$$
$$\mathbf{n} \rightarrow \mathbf{n}, \tag{A.78}$$
$$\mathbf{m} \rightarrow c\mathbf{n} + \mathbf{m}, \tag{A.79}$$
$$\bar{\mathbf{m}} \rightarrow \bar{c}\mathbf{n} + \bar{\mathbf{m}}, \tag{A.80}$$
$$\tag{A.81}$$

Here again, c is a complex constant. Accordingly the spin coefficients transform as

$$
\begin{aligned}
\nu &\rightarrow \nu, & \text{(A.82)}\\
\tau &\rightarrow \tau + 2c\gamma + c^2\nu - D_{\mathbf{n}}c, & \text{(A.83)}\\
\gamma &\rightarrow \gamma + c\nu, & \text{(A.84)}\\
\mu &\rightarrow \mu + c\nu, & \text{(A.85)}\\
\sigma &\rightarrow \sigma + c\tau + 2c\beta + 2c^2\gamma + c^2\mu + c^3\nu - D_{\mathbf{m}}c - cD_{\mathbf{n}}c & \text{(A.86)}\\
\beta &\rightarrow \beta + c\gamma + c\mu + c^2\nu, & \text{(A.87)}\\
\lambda &\rightarrow \lambda + \bar{c}\nu, & \text{(A.88)}\\
\rho &\rightarrow +\rho + \bar{c}\tau + c^2\lambda + c^2\bar{c}\nu + 2c\alpha + 2c\bar{c}\gamma - D_{\overline{\mathbf{m}}}c - \bar{c}D_{\mathbf{n}}c, & \text{(A.89)}\\
\alpha &\rightarrow \alpha + \bar{c}\tau + c\lambda + c\bar{c}\nu, & \text{(A.90)}\\
\kappa &\rightarrow \kappa + c\bar{c}\tau + \bar{c}\sigma + c\rho + c^2\pi + c^3\bar{c}\nu + c^2\bar{c}\mu + c^3\lambda + c\varepsilon + 2c^2\bar{c}\gamma + 2c\bar{c}\beta + 2c^2\alpha & \\
&- D_{\mathbf{l}}c - c\bar{c}D_{\mathbf{n}}c - \bar{c}D_{\mathbf{m}}c - cD_{\overline{\mathbf{m}}}c, & \text{(A.91)}\\
\varepsilon &\rightarrow \varepsilon + c\bar{c}\gamma + \bar{c}\beta + c\alpha + c\pi + c^2\bar{c}\nu + c\bar{c}\mu + c^2\lambda, & \text{(A.92)}\\
\pi &\rightarrow \pi + c\lambda + \bar{c}\mu + c\bar{c}\nu. & \text{(A.93)}
\end{aligned}
$$

The tranformations of Ricci scalars are given by

$$
\begin{aligned}
\Phi_{00} &\rightarrow \Phi_{00} + 4c\bar{c}\Phi_{11} + 2\bar{c}\Phi_{01} + 2c\Phi_{10} + c^2\bar{c}^2\Phi_{22} + 2c\bar{c}^2\Phi_{12} & \\
&+ 2c^2\bar{c}\Phi_{21} + c^2\Phi_{20} + \bar{c}^2\Phi_{02}, & \text{(A.94)}\\
\Phi_{01} &\rightarrow \Phi_{01} + 2c\bar{c}\Phi_{12} + c^2\Phi_{21} + \bar{c}\Phi_{02} + 2c\Phi_{11} + c^2\bar{c}\Phi_{22}, & \text{(A.95)}\\
\Phi_{10} &\rightarrow \Phi_{10} + 2c\bar{c}\Phi_{21} + \bar{c}\Phi_{12} + c\Phi_{20} + 2\bar{c}\Phi_{11} + \bar{c}^2c\Phi_{22}, & \text{(A.96)}\\
\Phi_{02} &\rightarrow \Phi_{02} + 2c\Phi_{12} + c^2\Phi_{22}, & \text{(A.97)}\\
\Phi_{20} &\rightarrow \Phi_{20} + 2\bar{c}\Phi_{21} + \bar{c}^2\Phi_{22}, & \text{(A.98)}\\
\Phi_{11} &\rightarrow \Phi_{11} + c\bar{c}\Phi_{22} + \bar{c}\Phi_{12} + c\Phi_{21}, & \text{(A.99)}\\
\Phi_{12} &\rightarrow \Phi_{12} + c\Phi_{22}, & \text{(A.100)}\\
\Phi_{21} &\rightarrow \Phi_{21} + \bar{c}\Phi_{22}, & \text{(A.101)}\\
\Phi_{22} &\rightarrow \Phi_{22}. & \text{(A.102)}
\end{aligned}
$$

and the tranformations of Weyl scalars are given by

$$\Psi_0 \rightarrow \Psi_0 + 4c\Psi_1 + 6c^2\Psi_2 + 4c^3\Psi_3 + c^4\Psi_4, \tag{A.103}$$
$$\Psi_1 \rightarrow \Psi_1 + 3c\Psi_2 + 3c^2\Psi_3 + c^3\Psi_4, \tag{A.104}$$
$$\Psi_2 \rightarrow \Psi_2 + 2c\Psi_3 + c^2\Psi_4, \tag{A.105}$$
$$\Psi_3 \rightarrow \Psi_3 + c\Psi_4, \tag{A.106}$$
$$\Psi_4 \rightarrow \Psi_4. \tag{A.107}$$

A.5 Type-III Lorentz transformations

Type-III Lorentz transformation represents the boosting of **l** and **n** and the rotation of **m** and $\overline{\mathbf{m}}$, i.e., the tetrad vectors transform as

$$\mathbf{l} \rightarrow a^2\mathbf{l}, \tag{A.108}$$
$$\mathbf{n} \rightarrow \frac{1}{a^2}\mathbf{n}, \tag{A.109}$$
$$\mathbf{m} \rightarrow e^{2i\theta}\mathbf{m}, \tag{A.110}$$
$$\overline{\mathbf{m}} \rightarrow e^{-2i\theta}\overline{\mathbf{m}}. \tag{A.111}$$

Here both a and θ are real constants. Accordingly the spin coefficients transform as

$$\nu \rightarrow a^{-4}e^{-2i\theta}\nu, \tag{A.112}$$
$$\tau \rightarrow e^{2i\theta}\tau, \tag{A.113}$$
$$\gamma \rightarrow a^{-2}(\gamma + D_{\mathbf{n}}[\ln a + i\theta]), \tag{A.114}$$
$$\mu \rightarrow a^{-2}\mu, \tag{A.115}$$
$$\sigma \rightarrow a^2e^{4i\theta}\sigma, \tag{A.116}$$
$$\beta \rightarrow e^{2i\theta}(\beta + D_{\mathbf{m}}[\ln a + i\theta]), \tag{A.117}$$
$$\lambda \rightarrow a^{-2}e^{-4i\theta}\lambda, \tag{A.118}$$
$$\rho \rightarrow a^2\rho, \tag{A.119}$$

$$\begin{aligned}
\alpha &\rightarrow e^{-2i\theta}\left(\alpha + D_{\bar{\mathbf{m}}}\left[\ln a + i\theta\right]\right), &\quad& (A.120)\\
\kappa &\rightarrow a^4 e^{2i\theta}\kappa, && (A.121)\\
\varepsilon &\rightarrow a^2\left(\varepsilon + D_{\mathbf{l}}\left[\ln a + i\theta\right]\right), && (A.122)\\
\pi &\rightarrow e^{-2i\theta}\pi. && (A.123)
\end{aligned}$$

The tranformations of Ricci scalars are given by

$$\begin{aligned}
\Phi_{00} &\rightarrow a^4\Phi_{00}, &\quad& (A.124)\\
\Phi_{01} &\rightarrow a^2 e^{2i\theta}\Phi_{01}, && (A.125)\\
\Phi_{10} &\rightarrow a^2 e^{-2i\theta}\Phi_{10}, && (A.126)\\
\Phi_{02} &\rightarrow e^{4i\theta}\Phi_{02}, && (A.127)\\
\Phi_{20} &\rightarrow e^{-4i\theta}\Phi_{20}, && (A.128)\\
\Phi_{11} &\rightarrow \Phi_{11}, && (A.129)\\
\Phi_{12} &\rightarrow a^{-2} e^{2i\theta}\Phi_{12}, && (A.130)\\
\Phi_{21} &\rightarrow a^{-2} e^{-2i\theta}\Phi_{21}, && (A.131)\\
\Phi_{22} &\rightarrow a^{-4}\Phi_{22}, && (A.132)
\end{aligned}$$

and the tranformations of Weyl scalars are given by

$$\begin{aligned}
\Psi_0 &\rightarrow a^4 e^{4i\theta}\Psi_0, &\quad& (A.133)\\
\Psi_1 &\rightarrow a^2 e^{2i\theta}\Psi_1, && (A.134)\\
\Psi_2 &\rightarrow \Psi_2, && (A.135)\\
\Psi_3 &\rightarrow a^{-2} e^{-2i\theta}\Psi_3, && (A.136)\\
\Psi_4 &\rightarrow a^{-4} e^{-4i\theta}\Psi_4. && (A.137)
\end{aligned}$$

B Raychaudhuri equation in Newman-Penrose formalism

B.1 Useful expressions

The following expressions are used many times in our transformation to the NP formalism

$$
\begin{aligned}
\eta^{ab} E^{\rho}{}_{b} E^{\gamma}{}_{a} &= -E^{\rho}{}_{\hat{0}} E^{\gamma}{}_{\hat{0}} + E^{\rho}{}_{\hat{1}} E^{\gamma}{}_{\hat{1}} \\
&= -\left(\frac{1}{\sqrt{2}}\right)^2 (l^{\rho}+n^{\rho})(l^{\gamma}+n^{\gamma}) + \left(\frac{1}{\sqrt{2}}\right)^2 (l^{\rho}-n^{\rho})(l^{\gamma}-n^{\gamma}) \\
&= -(l^{\rho} n^{\gamma} + l^{\gamma} n^{\rho}). \qquad \text{(B.1)} \\
\delta^{ij} N^{\nu}{}_{i} N^{\beta}{}_{j} &= N^{\nu}{}_{\hat{2}} N^{\beta}{}_{\hat{2}} + N^{\nu}{}_{\hat{3}} N^{\beta}{}_{\hat{3}} \\
&= \left(\frac{1}{\sqrt{2}}\right)^2 (m^{\nu}+\overline{m}^{\nu})\left(m^{\beta}+\overline{m}^{\beta}\right) \\
&+ \left(\frac{-i}{\sqrt{2}}\right)^2 (m^{\nu}-\overline{m}^{\nu})\left(m^{\beta}-\overline{m}^{\beta}\right) \\
&= \left(m^{\nu}\overline{m}^{\beta} + m^{\beta}\overline{m}^{\nu}\right). \qquad \text{(B.2)} \\
\eta^{ab} E^{\beta}{}_{a} D_{\alpha} E^{\mu}{}_{b} &= -E^{\beta}{}_{\hat{0}} D_{\mu} E^{\beta}{}_{\hat{0}} + E^{\mu}{}_{\hat{1}} D_{\alpha} E^{\rho}{}_{\hat{1}} \\
&= -\frac{1}{2}\left(l^{\beta}+n^{\beta}\right) D_{\alpha}(l^{\mu}+n^{\mu}) + \frac{1}{2}\left(l^{\beta}-n^{\beta}\right) D_{\alpha}(l^{\mu}-n^{\mu}) \\
&= -\left(l^{\beta} D_{\alpha} n^{\mu} + n^{\beta} D_{\alpha} l^{\mu}\right). \qquad \text{(B.3)} \\
\eta^{ab} E^{\alpha}{}_{a} D_{\alpha} E^{\mu}{}_{b} &= -(D_{\mathbf{l}} n^{\mu} + D_{\mathbf{n}} l^{\mu}), \qquad \text{(B.4)} \\
\delta^{ij} N^{\alpha}{}_{i} D_{\beta} N^{\nu}{}_{j} &= N^{\alpha}{}_{\hat{2}} D_{\beta} N^{\nu}{}_{\hat{2}} + N^{\alpha}{}_{\hat{3}} D_{\beta} N^{\nu}{}_{\hat{3}} \\
&= \frac{1}{2}(m^{\alpha}+\overline{m}^{\alpha}) D_{\beta}(m^{\nu}+\overline{m}^{\nu}) - \frac{1}{2}(m^{\alpha}-\overline{m}^{\alpha}) D_{\beta}(m^{\nu}-\overline{m}^{\nu}) \\
&= m^{\alpha} D_{\beta}\overline{m}^{\nu} + \overline{m}^{\alpha} D_{\beta} m^{\nu}, \qquad \text{(B.5)} \\
\delta^{ij} N^{\alpha}{}_{i} D_{\alpha} N^{\nu}{}_{j} &= D_{\mathbf{m}}\overline{m}^{\nu} + D_{\overline{\mathbf{m}}} m^{\nu}. \qquad \text{(B.6)}
\end{aligned}
$$

$$
\begin{aligned}
\eta^{cd}\left(D_\rho E^\mu{}_c\right)\left(D_\gamma E^\alpha{}_d\right) &= -\left(D_\rho E^\mu{}_{\hat{0}}\right)\left(D_\gamma E^\alpha{}_{\hat{0}}\right)+\left(D_\rho E^\mu{}_{\hat{1}}\right)\left(D_\gamma E^\alpha{}_{\hat{1}}\right) \\
&= -\frac{1}{2}\left(D_\rho l^\mu + D_\rho n^\mu\right)\left(D_\gamma l^\alpha + D_\gamma n^\alpha\right) \qquad \text{(B.7)} \\
&+ \frac{1}{2}\left(D_\rho l^\mu - D_\rho n^\mu\right)\left(D_\gamma l^\alpha - D_\gamma n^\alpha\right) \\
&= -\left[\left(D_\rho l^\mu\right)\left(D_\gamma n^\alpha\right)+\left(D_\rho n^\mu\right)\left(D_\gamma l^\alpha\right)\right]. \qquad \text{(B.8)} \\
\eta^{ab}E^\beta{}_b D_\beta D_\gamma E^\mu{}_a &= -E^\beta{}_{\hat{0}} D_\beta D_\gamma E^\mu{}_{\hat{0}} + E^\beta{}_{\hat{1}} D_\beta D_\gamma E^\mu{}_{\hat{1}} \\
&= -\frac{1}{2}\left(l^\beta + n^\beta\right)D_\beta D_\gamma\left(l^\mu + n^\mu\right)+\frac{1}{2}\left(l^\beta - n^\beta\right)D_\beta D_\gamma\left(l^\mu - n^\mu\right) \\
&= -\frac{1}{2}\left[D_{\mathbf{l}} D_\gamma\left(l^\mu + n^\nu\right)+D_{\mathbf{n}} D_\gamma\left(l^\mu + n^\nu\right)\right] \\
&+ \frac{1}{2}\left[D_{\mathbf{l}} D_\gamma\left(l^\mu - n^\nu\right)-D_{\mathbf{n}} D_\gamma\left(l^\mu - n^\nu\right)\right] \\
&= -\left(D_{\mathbf{l}} D_\gamma n^\mu + D_{\mathbf{n}} D_\gamma l^\mu\right). \qquad \text{(B.9)}
\end{aligned}
$$

B.2 Derivation of $\tilde{\nabla}_{\mathbb{T}}\mathcal{J}$

Consider the left hand side of the Raychaudhuri equation (4.43), and the worldsheet covariant derivative of J_{aij} defined in relation (2.97), i.e.,

$$
\tilde{\nabla}_{\mathbb{T}}\mathcal{J} = \eta^{ab}\delta^{ij}\tilde{\nabla}_b J_{aij} = \eta^{ab}\delta^{ij}\left(\underbrace{\nabla_b J_{aij}}_{D_b J_{aij} - \gamma_{ba}{}^c J_{cij}} - w_{bi}{}^k J_{akj} - w_{bj}{}^k J_{aik}\right). \qquad \text{(B.10)}
$$

By using the definition of J_{aij}, eq. (2.77), the first term of the equation (B.10) becomes,

$$
\begin{aligned}
\eta^{ab}\delta^{ij}D_b J_{aij} &:= \eta^{ab}\delta^{ij}D_b\left[g_{\mu\nu}D_i\left(E^\mu{}_a\right)N^\nu{}_j\right] = g_{\mu\nu}\eta^{ab}\delta^{ij}\left(D_b N^\gamma{}_i\right)\left(D_\gamma E^\mu{}_a\right)N^\nu{}_j \\
&+ g_{\mu\nu}\eta^{ab}\delta^{ij}N^\gamma{}_i\left(D_b D_\gamma E^\mu{}_a\right)N^\nu{}_j + g_{\mu\nu}\eta^{ab}\delta^{ij}N^\gamma{}_i\left(D_\gamma E^\mu{}_a\right)E^\beta{}_b\left(D_\beta N^\nu{}_j\right) \\
&= g_{\mu\nu}\left(\delta^{ij}N^\nu{}_j D_\beta N^\gamma{}_i\right)\left(\eta^{ab}E^\beta{}_b D_\gamma E^\mu{}_a\right) \\
&+ g_{\mu\nu}\left(\delta^{ij}N^\gamma{}_i N^\nu{}_j\right)\left(\eta^{ab}E^\beta{}_b D_\beta D_\gamma E^\mu{}_a\right) \\
&+ g_{\mu\nu}\left(\delta^{ij}N^\gamma{}_i D_\beta N^\nu{}_j\right)\left(\eta^{ab}E^\beta{}_b D_\gamma E^\mu{}_a\right),
\end{aligned}
$$

and by making use of eqs. (B.2), (B.3), (B.5) and (B.9),

$$
\begin{aligned}
\eta^{ab}\delta^{ij}D_b J_{aij} &= -g_{\mu\nu}\left(m^\nu D_\beta \overline{m}^\gamma + \overline{m}^\nu D_\beta m^\gamma\right)\left(l^\beta D_\gamma n^\mu + n^\beta D_\gamma l^\mu\right) \\
&- \left(m^\gamma \overline{m}^\nu + m^\nu \overline{m}^\gamma\right)\left(D_{\mathbf{l}} D_\gamma n^\mu + D_{\mathbf{n}} D_\gamma l^\mu\right) \\
&- g_{\mu\nu}\left(m^\gamma D_\beta \overline{m}^\nu + \overline{m}^\gamma D_\beta m^\nu\right)\left(l^\beta D_\gamma n^\mu + n^\beta D_\gamma l^\mu\right) \\
&= -g_{\mu\nu}\left[\overline{m}^\nu\left(D_\beta m^\gamma\right) l^\beta\left(D_\gamma n^\mu\right) + \overline{m}^\nu m^\gamma D_{\mathbf{l}} D_\gamma n^\mu\right] \\
&- g_{\mu\nu}\left[m^\nu\left(D_\beta \overline{m}^\gamma\right) l^\beta\left(D_\gamma n^\mu\right) + m^\nu \overline{m}^\gamma D_{\mathbf{l}} D_\gamma n^\mu\right] \\
&- g_{\mu\nu}\left[\overline{m}^\nu\left(D_\beta m^\gamma\right) n^\beta\left(D_\gamma l^\mu\right) + \overline{m}^\nu m^\gamma D_{\mathbf{n}} D_\gamma l^\mu\right] \\
&- g_{\mu\nu}\left[m^\nu\left(D_\beta \overline{m}^\gamma\right) n^\beta\left(D_\gamma l^\mu\right) + m^\nu \overline{m}^\gamma D_{\mathbf{n}} D_\gamma l^\mu\right] \\
&- g_{\mu\nu}\left[(D_{\mathbf{l}}\overline{m}^\nu)(D_{\mathbf{m}} n^\mu) + (D_{\mathbf{n}}\overline{m}^\nu)(D_{\mathbf{m}} l^\mu)\right] \\
&- g_{\mu\nu}\left[(D_{\mathbf{l}} m^\nu)(D_{\overline{\mathbf{m}}} n^\mu) + (D_{\mathbf{n}} m^\nu)(D_{\overline{\mathbf{m}}} l^\mu)\right] \\
&= -\left[\langle \overline{\mathbf{m}}, D_{\mathbf{l}} D_{\mathbf{m}} \mathbf{n}\rangle + \langle \mathbf{m}, D_{\mathbf{l}} D_{\overline{\mathbf{m}}} \mathbf{n}\rangle\right] - \left[\langle \overline{\mathbf{m}}, D_{\mathbf{n}} D_{\mathbf{m}} \mathbf{l}\rangle + \langle \mathbf{m}, D_{\mathbf{n}} D_{\overline{\mathbf{m}}} \mathbf{l}\rangle\right] \\
&- \left[\langle D_{\mathbf{l}}\overline{\mathbf{m}}, D_{\mathbf{m}} \mathbf{n}\rangle + \langle D_{\mathbf{n}}\overline{\mathbf{m}}, D_{\mathbf{m}} \mathbf{l}\rangle\right] - \left[\langle D_{\mathbf{l}} \mathbf{m}, D_{\overline{\mathbf{m}}} \mathbf{n}\rangle + \langle D_{\mathbf{n}} \mathbf{m}, D_{\overline{\mathbf{m}}} \mathbf{l}\rangle\right].
\end{aligned}
$$

Now we can use eqs. (A.11), (A.12), (A.14), (A.15) and (A.16) to obtain

$$
\begin{aligned}
\eta^{ab}\delta^{ij}D_b J_{aij} &= \left[D_{\mathbf{n}}(\rho+\overline{\rho}) - D_{\mathbf{l}}(\mu+\overline{\mu})\right] + \left[(\overline{\alpha}+\beta)(\pi+\overline{\tau}) + \left(\alpha+\overline{\beta}\right)(\overline{\pi}+\tau)\right] \\
&- \left[(\varepsilon-\overline{\varepsilon})(\mu-\overline{\mu}) + (\gamma-\overline{\gamma})(\rho-\overline{\rho})\right] - \left[(\overline{\alpha}+\beta)(\pi+\overline{\tau}) + \left(\alpha+\overline{\beta}\right)(\overline{\pi}+\tau)\right] \\
&+ \left[(\varepsilon-\overline{\varepsilon})(\mu-\overline{\mu}) + (\gamma-\overline{\gamma})(\rho-\overline{\rho})\right] \\
&= \left[D_{\mathbf{n}}(\rho+\overline{\rho}) - D_{\mathbf{l}}(\mu+\overline{\mu})\right].
\end{aligned} \tag{B.11}
$$

In order to derive the second term of eq. (B.10), we will use the definitions in eq. (2.75) and eq. (2.77). Then we get

$$
\begin{aligned}
\eta^{ab}\delta^{ij}\gamma_{ba}{}^{c} J_{cij} &= \eta^{ab}\eta^{cd}\delta^{ij}\left(g_{\mu\nu}\left[D_b E^\mu{}_a\right] E^\nu{}_d\right)\left(g_{\alpha\beta}\left[D_i E^\alpha{}_c\right] N^\beta{}_j\right) \\
&= g_{\mu\nu} g_{\alpha\beta}\left(N^\gamma{}_i N^\beta{}_j \delta^{ij}\right)\left(\eta^{ab} E^\rho{}_b D_\rho E^\mu{}_a\right)\left(\eta^{cd} E^\nu{}_d D_\gamma E^\alpha{}_c\right).
\end{aligned}
$$

Then by using relations (B.2), (B.3) and (B.4),

$$
\begin{aligned}
\eta^{ab}\delta^{ij}\gamma_{ba}{}^{c} J_{cij} &= g_{\mu\nu} g_{\alpha\beta}\left(m^\gamma \overline{m}^\beta + m^\beta \overline{m}^\gamma\right)\left(D_{\mathbf{l}} n^\mu + D_{\mathbf{n}} l^\mu\right)\left(l^\nu D_\gamma n^\alpha + n^\nu D_\gamma l^\alpha\right) \\
&= g_{\mu\nu} g_{\alpha\beta}\left(D_{\mathbf{l}} n^\mu + D_{\mathbf{n}} l^\mu\right) \\
&\times \left(\overline{m}^\beta l^\nu D_{\mathbf{m}} n^\alpha + \overline{m}^\beta n^\nu D_{\mathbf{m}} l^\alpha + m^\beta l^\nu D_{\overline{\mathbf{m}}} n^\alpha + m^\beta n^\nu D_{\overline{\mathbf{m}}} l^\alpha\right) \\
&= \langle D_{\mathbf{m}}\mathbf{n}, \overline{\mathbf{m}}\rangle\left(\langle D_{\mathbf{l}}\mathbf{n}, \mathbf{l}\rangle + \langle D_{\mathbf{n}}\mathbf{l}, \mathbf{l}\rangle\right) + \langle D_{\mathbf{m}}\mathbf{l}, \overline{\mathbf{m}}\rangle\left(\langle D_{\mathbf{l}}\mathbf{n}, \mathbf{n}\rangle + \langle D_{\mathbf{n}}\mathbf{l}, \mathbf{n}\rangle\right) \\
&+ \langle D_{\overline{\mathbf{m}}}\mathbf{n}, \mathbf{m}\rangle\left(\langle D_{\mathbf{l}}\mathbf{n}, \mathbf{l}\rangle + \langle D_{\mathbf{n}}\mathbf{l}, \mathbf{l}\rangle\right) + \langle D_{\overline{\mathbf{m}}}\mathbf{l}, \mathbf{m}\rangle\left(\langle D_{\mathbf{l}}\mathbf{n}, \mathbf{n}\rangle + \langle D_{\mathbf{n}}\mathbf{l}, \mathbf{n}\rangle\right),
\end{aligned}
$$

and by using eqs. (A.10), (A.11), (A.12) and (A.14) we obtain

$$\eta^{ab}\delta^{ij}\gamma_{ba}{}^{c}J_{cij} = (\varepsilon+\overline{\varepsilon})(\mu+\overline{\mu})+(\gamma+\overline{\gamma})(\rho+\overline{\rho}). \tag{B.12}$$

In order to derive the third term of eq. (B.10) one uses the definitions in eq. (2.76) and eq. (2.77). Then we write

$$\begin{aligned}\eta^{ab}\delta^{ij}w_{bi}{}^{k}J_{akj} &= \eta^{ab}\delta^{ij}\delta^{kl}\left(g_{\mu\nu}\left[D_b N^{\mu}{}_{i}\right]N^{\nu}{}_{k}\right)\left(g_{\alpha\beta}\left[D_l E^{\alpha}{}_{a}\right]N^{\beta}{}_{j}\right)\\ &= g_{\mu\nu}g_{\alpha\beta}\left(\delta^{kl}N^{\gamma}{}_{l}N^{\nu}{}_{k}\right)\left(\delta^{ij}N^{\beta}{}_{j}D_{\rho}N^{\mu}{}_{i}\right)\left(\eta^{ab}E^{\rho}{}_{b}D_{\gamma}E^{\alpha}{}_{a}\right).\end{aligned}$$

Now using eqs. (B.2), (B.3) and (B.5) results in

$$\begin{aligned}\eta^{ab}\delta^{ij}w_{bj}{}^{k}J_{aki} &= -g_{\mu\nu}g_{\alpha\beta}(m^{\gamma}\overline{m}^{\nu}+m^{\nu}\overline{m}^{\gamma})\\ &\times \left(m^{\beta}D_{\rho}\overline{m}^{\mu}+\overline{m}^{\beta}D_{\rho}m^{\mu}\right)\left(l^{\rho}D_{\gamma}n^{\alpha}+n^{\rho}D_{\gamma}l^{\alpha}\right)\\ &= -[\langle D_{\mathbf{m}}\mathbf{n},\mathbf{m}\rangle\langle D_{\mathbf{l}}\overline{\mathbf{m}},\overline{\mathbf{m}}\rangle+\langle D_{\mathbf{l}}\mathbf{m},\overline{\mathbf{m}}\rangle\langle D_{\mathbf{m}}\mathbf{n},\overline{\mathbf{m}}\rangle\\ &+ \langle D_{\mathbf{n}}\overline{\mathbf{m}},\overline{\mathbf{m}}\rangle\langle D_{\mathbf{m}}\mathbf{l},\mathbf{m}\rangle+\langle D_{\mathbf{n}}\mathbf{m},\overline{\mathbf{m}}\rangle\langle D_{\mathbf{m}}\mathbf{l},\overline{\mathbf{m}}\rangle\\ &+ \langle D_{\mathbf{l}}\mathbf{m},\mathbf{m}\rangle\langle D_{\overline{\mathbf{m}}}\mathbf{n},\mathbf{m}\rangle+\langle D_{\mathbf{l}}\mathbf{m},\mathbf{m}\rangle\langle D_{\overline{\mathbf{m}}}\mathbf{n},\overline{\mathbf{m}}\rangle\\ &+ \langle D_{\mathbf{n}}\overline{\mathbf{m}},\mathbf{m}\rangle\langle D_{\overline{\mathbf{m}}}\mathbf{l},\mathbf{m}\rangle+\langle D_{\mathbf{n}}\mathbf{m},\mathbf{m}\rangle\langle D_{\overline{\mathbf{m}}}\mathbf{l},\overline{\mathbf{m}}\rangle],\end{aligned}$$

and by eqs. (A.11), (A.14), (A.15) and (A.16) we obtain

$$\eta^{ab}\delta^{ij}w_{bj}{}^{k}J_{aki} = -([\varepsilon-\overline{\varepsilon}][\mu-\overline{\mu}]+[\gamma-\overline{\gamma}][\rho-\overline{\rho}]). \tag{B.13}$$

Similarly, the fourth term in eq. (B.10) follows from

$$\begin{aligned}\eta^{ab}\delta^{ij}w_{bj}{}^{k}J_{aik} &= \eta^{ab}\delta^{ij}\delta^{kl}\left(g_{\mu\nu}\left[D_b N^{\mu}{}_{j}\right]N^{\nu}{}_{k}\right)\left(g_{\alpha\beta}\left[D_i E^{\alpha}{}_{a}\right]N^{\beta}{}_{l}\right)\\ &= g_{\mu\nu}g_{\alpha\beta}\left(\delta^{kl}N^{\nu}{}_{k}N^{\beta}{}_{l}\right)\left(\delta^{ij}N^{\gamma}{}_{i}D_{\rho}N^{\mu}{}_{j}\right)\left(\eta^{ab}E^{\rho}{}_{b}D_{\gamma}E^{\alpha}{}_{a}\right).\end{aligned}$$

Then by using relations (B.2), (B.3) and (B.5),

$$\begin{aligned}\eta^{ab}\delta^{ij}w_{bj}{}^{k}J_{aik} &= -g_{\mu\nu}g_{\alpha\beta}\left(m^{\nu}\overline{m}^{\beta}+m^{\beta}\overline{m}^{\nu}\right)\\ &\times \left(m^{\gamma}D_{\rho}\overline{m}^{\mu}+\overline{m}^{\gamma}D_{\rho}m^{\mu}\right)\left(l^{\rho}D_{\gamma}n^{\alpha}+n^{\rho}D_{\gamma}l^{\alpha}\right)\\ &= -[\langle D_{\mathbf{m}}\mathbf{n},\mathbf{m}\rangle\langle D_{\mathbf{l}}\overline{\mathbf{m}},\overline{\mathbf{m}}\rangle+\langle D_{\mathbf{l}}\overline{\mathbf{m}},\mathbf{m}\rangle\langle D_{\mathbf{m}}\mathbf{n},\overline{\mathbf{m}}\rangle\\ &+ \langle D_{\mathbf{n}}\overline{\mathbf{m}},\mathbf{m}\rangle\langle D_{\mathbf{m}}\mathbf{l},\overline{\mathbf{m}}\rangle+\langle D_{\mathbf{n}}\mathbf{m},\mathbf{m}\rangle\langle D_{\overline{\mathbf{m}}}\mathbf{l},\overline{\mathbf{m}}\rangle\\ &+ \langle D_{\mathbf{l}}\mathbf{m},\overline{\mathbf{m}}\rangle\langle D_{\overline{\mathbf{m}}}\mathbf{n},\mathbf{m}\rangle+\langle D_{\mathbf{l}}\mathbf{m},\mathbf{m}\rangle\langle D_{\overline{\mathbf{m}}}\mathbf{n},\overline{\mathbf{m}}\rangle\\ &+ \langle D_{\mathbf{n}}\overline{\mathbf{m}},\overline{\mathbf{m}}\rangle\langle D_{\mathbf{m}}\mathbf{l},\mathbf{m}\rangle+\langle D_{\mathbf{n}}\mathbf{m},\overline{\mathbf{m}}\rangle\langle D_{\overline{\mathbf{m}}}\mathbf{l},\mathbf{m}\rangle],\end{aligned}$$

and by further using eqs. (A.11), (A.14), (A.15) and (A.16) we obtain the same result as in (B.13), i.e.,

$$\eta^{ab}\delta^{ij} w_{bj}{}^{k} J_{aik} = -([\varepsilon-\overline{\varepsilon}][\mu-\overline{\mu}]+[\gamma-\overline{\gamma}][\rho-\overline{\rho}]). \tag{B.14}$$

Hence, substitution of the relations (B.11), (B.12), (B.13) and (B.14) into eq. (B.10) results in

$$\begin{aligned}\eta^{ab}\delta^{ij}\tilde{\nabla}_b J_{aij} &= [D_{\mathbf{n}}(\rho+\overline{\rho}) - D_{\mathbf{l}}(\mu+\overline{\mu})] - [(\varepsilon+\overline{\varepsilon})(\mu+\overline{\mu}) + (\gamma+\overline{\gamma})(\rho+\overline{\rho})] \\ &+ 2[(\varepsilon-\overline{\varepsilon})(\mu-\overline{\mu}) + (\gamma-\overline{\gamma})(\rho-\overline{\rho})].\end{aligned} \tag{B.15}$$

B.3 Derivation of $\tilde{\nabla}_{\mathbb{S}}\mathcal{K}$

Consider the first term on the right hand side of the Raychaudhuri equation (4.43), and the covariant derivative of K_{abj} on the spacelike 2-surface defined in relation (2.110), i.e.,

$$\tilde{\nabla}_{\mathbb{S}}\mathcal{K} := \eta^{ab}\delta^{ij}\tilde{\nabla}_i K_{abj} = \eta^{ab}\delta^{ij}\left(\underbrace{\nabla_i K_{abj}}_{D_i K_{abj} - \gamma_{ijk} K_{ab}{}^{k}} - S_{aci} K_{b}{}^{c}{}_{j} - S_{bci} K_{a}{}^{c}{}_{j}\right). \tag{B.16}$$

Then, by making use of the definition (2.74), the first term of eq. (B.16) is as follows

$$\begin{aligned}D_i K_{abj}\eta^{ab}\delta^{ij} &= \eta^{ab}\delta^{ij} D_i\left[-g_{\mu\nu}\left(D_a E^{\mu}{}_{b}\right) N^{\nu}{}_{j}\right] \\ &= -\eta^{ab}\delta^{ij}\left[N^{\nu}{}_{j} N^{\gamma}{}_{i} D_{\gamma}\left(g_{\mu\nu}\left(D_a E^{\mu}{}_{b}\right)\right)\right] - \eta^{ab}\delta^{ij}\left[g_{\mu\nu}\left(D_a E^{\mu}{}_{b}\right) N^{\gamma}{}_{i} D_{\gamma} N^{\nu}{}_{j}\right] \\ &= -g_{\mu\nu}\left[\left(\delta^{ij} N^{\nu}{}_{j} N^{\gamma}{}_{j}\right)\eta^{ab} D_{\gamma}\left(E^{\beta}{}_{a} D_{\beta} E^{\mu}{}_{b}\right)\right] \\ &- g_{\mu\nu}\left[\left(\eta^{ab} E^{\beta}{}_{a} D_{\beta} E^{\mu}{}_{b}\right)\left(\delta^{ij} N^{\gamma}{}_{i} D_{\gamma} N^{\nu}{}_{j}\right)\right].\end{aligned}$$

By using eqs. (B.2), (B.4) and (B.6) we write

$$\begin{aligned}D_i K_{abj}\eta^{ab}\delta^{ij} &= g_{\mu\nu}\left[(m^{\gamma}\overline{m}^{\nu} + m^{\nu}\overline{m}^{\gamma}) D_{\gamma}(D_{\mathbf{l}} n^{\mu} + D_{\mathbf{n}} l^{\mu})\right] \\ &+ g_{\mu\nu}[(D_{\mathbf{l}} n^{\mu} + D_{\mathbf{n}} l^{\mu})(D_{\mathbf{m}}\overline{m}^{\nu} + D_{\overline{\mathbf{m}}} m^{\nu})] \\ &= \langle\overline{\mathbf{m}}, D_{\mathbf{m}} D_{\mathbf{l}}\mathbf{n}\rangle + \langle\overline{\mathbf{m}}, D_{\mathbf{m}} D_{\mathbf{n}}\mathbf{l}\rangle + \langle\mathbf{m}, D_{\overline{\mathbf{m}}} D_{\mathbf{l}}\mathbf{n}\rangle + \langle\mathbf{m}, D_{\overline{\mathbf{m}}} D_{\mathbf{n}}\mathbf{l}\rangle \\ &+ \langle D_{\mathbf{l}}\mathbf{n}, D_{\mathbf{m}}\overline{\mathbf{m}}\rangle + \langle D_{\mathbf{l}}\mathbf{n}, D_{\overline{\mathbf{m}}}\mathbf{m}\rangle + \langle D_{\mathbf{n}}\mathbf{l}, D_{\mathbf{m}}\overline{\mathbf{m}}\rangle + \langle D_{\mathbf{n}}\mathbf{l}, D_{\overline{\mathbf{m}}}\mathbf{m}\rangle,\end{aligned}$$

and by further using eqs. (A.10), (A.12) and (A.18) we obtain

$$\begin{aligned} D_i K_{abj}\eta^{ab}\delta^{ij} &= D_{\mathbf{m}}(\pi-\overline{\tau})+D_{\overline{\mathbf{m}}}(\overline{\pi}-\tau)-\Big[\big(\alpha-\overline{\beta}\big)(\overline{\pi}-\tau)+(\overline{\alpha}-\beta)(\pi-\overline{\tau})\Big] \\ &- \big[(\varepsilon+\overline{\varepsilon})(\mu+\overline{\mu})+(\gamma+\overline{\gamma})(\rho+\overline{\rho})\big]+\big[(\varepsilon+\overline{\varepsilon})(\mu+\overline{\mu})+(\gamma+\overline{\gamma})(\rho+\overline{\rho})\big] \\ &+ \Big[\big(\alpha-\overline{\beta}\big)(\overline{\pi}-\tau)+(\overline{\alpha}-\beta)(\pi-\overline{\tau})\Big] \\ &= D_{\mathbf{m}}(\pi-\overline{\tau})+D_{\overline{\mathbf{m}}}(\overline{\pi}-\tau). \end{aligned} \qquad \text{(B.17)}$$

The second term in eq. (B.16) is obtained by using the definitions (2.74) and (2.78). The derivation follows as

$$\begin{aligned} \gamma_{ijk}K_{abl}\delta^{ij}\delta^{kl}\eta^{ab} &= \Big[g_{\alpha\beta}\big(D_i N^{\alpha}{}_j\big)N^{\beta}{}_k\Big]\Big[-g_{\mu\nu}\big(D_a E^{\mu}{}_b\big)N^{\nu}{}_l\Big]\delta^{ij}\delta^{kl}\eta^{ab} \\ &= -g_{\alpha\beta}g_{\mu\nu}\big(\delta^{kl}N^{\beta}{}_k N^{\nu}{}_l\big)\big(\delta^{ij}N^{\rho}{}_i D_\rho N^{\alpha}{}_j\big)\big(\eta^{ab}E^{\gamma}{}_a D_\gamma E^{\mu}{}_b\big). \end{aligned}$$

Now let us use eqs. (B.2), (B.4) and (B.5) to write

$$\begin{aligned} \gamma_{ijk}K_{abl}\delta^{ij}\delta^{kl}\eta^{ab} &= g_{\alpha\beta}g_{\mu\nu}\big(m^{\beta}\overline{m}^{\nu}+m^{\nu}\overline{m}^{\beta}\big)\big(m^{\rho}D_\rho\overline{m}^{\alpha}+\overline{m}^{\rho}D_\rho m^{\alpha}\big) \\ &\times \big(D_{\mathbf{l}}n^{\mu}+D_{\mathbf{n}}l^{\mu}\big) \\ &= \langle D_{\mathbf{m}}\overline{\mathbf{m}},\mathbf{m}\rangle\langle D_{\mathbf{l}}\mathbf{n},\overline{\mathbf{m}}\rangle+\langle D_{\overline{\mathbf{m}}}\mathbf{m},\mathbf{m}\rangle\langle D_{\mathbf{l}}\mathbf{n},\overline{\mathbf{m}}\rangle \\ &+ \langle D_{\mathbf{m}}\overline{\mathbf{m}},\mathbf{m}\rangle\langle D_{\mathbf{n}}\mathbf{l},\overline{\mathbf{m}}\rangle+\langle D_{\overline{\mathbf{m}}}\mathbf{m},\mathbf{m}\rangle\langle D_{\mathbf{n}}\mathbf{l},\overline{\mathbf{m}}\rangle \\ &+ \langle D_{\mathbf{m}}\overline{\mathbf{m}},\overline{\mathbf{m}}\rangle\langle D_{\mathbf{l}}\mathbf{n},\mathbf{m}\rangle+\langle D_{\overline{\mathbf{m}}}\mathbf{m},\overline{\mathbf{m}}\rangle\langle D_{\mathbf{l}}\mathbf{n},\mathbf{m}\rangle \\ &+ \langle D_{\mathbf{m}}\overline{\mathbf{m}},\overline{\mathbf{m}}\rangle\langle D_{\mathbf{n}}\mathbf{l},\mathbf{m}\rangle+\langle D_{\overline{\mathbf{m}}}\mathbf{m},\overline{\mathbf{m}}\rangle\langle D_{\mathbf{n}}\mathbf{l},\mathbf{m}\rangle. \end{aligned}$$

Then by using eqs. (A.10), (A.12) and (A.18) we obtain

$$\gamma_{ijk}K_{abl}\delta^{ij}\delta^{kl}\eta^{ab} = (\overline{\alpha}-\beta)(\pi-\overline{\tau})+\big(\alpha-\overline{\beta}\big)(\overline{\pi}-\tau). \qquad \text{(B.18)}$$

Finally we derive the third term that appears in eq. (B.16). Note that the third term is equal to the fourth term since our η_{ab} is diagonal. Here we make use of the definitions (2.79) and (2.74) and get

$$\begin{aligned} S_{aci}K_{bdj}\delta^{ij}\eta^{ab}\eta^{cd} &= \Big[g_{\mu\nu}\big(D_i E^{\mu}{}_a\big)E^{\nu}{}_c\Big]\Big[-g_{\alpha\beta}\big(D_b E^{\alpha}{}_d\big)N^{\beta}{}_j\Big]\delta^{ij}\eta^{ab}\eta^{cd} \\ &= -g_{\mu\nu}g_{\alpha\beta}\big(\delta^{ij}N^{\gamma}{}_i N^{\beta}{}_j\big)\big(\eta^{ab}E^{\rho}{}_b D_\gamma E^{\mu}{}_a\big)\big(\eta^{cd}E^{\nu}{}_c D_\rho E^{\alpha}{}_d\big). \end{aligned}$$

Also by using eqs. (B.2) and (B.3)

$$
\begin{aligned}
S_{aci}K_{bdj}\delta^{ij}\eta^{ab}\eta^{cd} &= -g_{\mu\nu}g_{\alpha\beta}\left(m^{\gamma}\overline{m}^{\beta}+m^{\beta}\overline{m}^{\gamma}\right)\left(l^{\rho}D_{\gamma}n^{\mu}+n^{\rho}D_{\gamma}l^{\mu}\right)\\
&\times \left(l^{\gamma}D_{\rho}n^{\alpha}+n^{\gamma}D_{\rho}l^{\alpha}\right)\\
&= -[\langle D_{\mathbf{m}}\mathbf{n},\mathbf{l}\rangle\langle D_{\mathbf{l}}\mathbf{n},\overline{\mathbf{m}}\rangle+\langle D_{\mathbf{m}}\mathbf{n},\mathbf{n}\rangle\langle D_{\mathbf{l}}\mathbf{l},\overline{\mathbf{m}}\rangle]\\
&= -[\langle D_{\mathbf{m}}\mathbf{l},\mathbf{l}\rangle\langle D_{\mathbf{n}}\mathbf{n},\overline{\mathbf{m}}\rangle+\langle D_{\mathbf{m}}\mathbf{l},\mathbf{n}\rangle\langle D_{\mathbf{n}}\mathbf{l},\overline{\mathbf{m}}\rangle]\\
&= -[\langle D_{\overline{\mathbf{m}}}\mathbf{n},\mathbf{l}\rangle\langle D_{\mathbf{l}}\mathbf{n},\mathbf{m}\rangle+\langle D_{\overline{\mathbf{m}}}\mathbf{n},\mathbf{n}\rangle\langle D_{\mathbf{l}}\mathbf{l},\mathbf{m}\rangle]\\
&= -[\langle D_{\overline{\mathbf{m}}}\mathbf{l},\mathbf{n}\rangle\langle D_{\mathbf{n}}\mathbf{l},\mathbf{m}\rangle+\langle D_{\overline{\mathbf{m}}}\mathbf{l},\mathbf{l}\rangle\langle D_{\mathbf{n}}\mathbf{n},\mathbf{m}\rangle].
\end{aligned}
$$

Then by further using eqs. (A.9), (A.10), (A.11), (A.12) and (A.13) we write

$$
S_{aci}K_{bdj}\delta^{ij}\eta^{ab}\eta^{cd} = -\left[(\overline{\alpha}+\beta)(\pi+\overline{\tau})+\left(\alpha+\overline{\beta}\right)(\overline{\pi}+\tau)\right]. \tag{B.19}
$$

Therefore substitution of relations (B.17), (B.18) and (B.19) into eq. (B.16) results in

$$
\begin{aligned}
\eta^{ab}\delta^{ij}\tilde{\nabla}_{i}K_{abj} &= D_{\mathbf{m}}(\pi-\overline{\tau})+D_{\overline{\mathbf{m}}}(\overline{\pi}-\tau)-\left[(\overline{\alpha}-\beta)(\pi-\overline{\tau})+\left(\alpha-\overline{\beta}\right)(\overline{\pi}-\tau)\right]\\
&+ 2\left[(\overline{\alpha}+\beta)(\pi+\overline{\tau})+\left(\alpha+\overline{\beta}\right)(\overline{\pi}+\tau)\right].
\end{aligned} \tag{B.20}
$$

B.4 Derivation of $\mathcal{J}^2$

In order to derive the second term that appears on the right hand side of the Raychaudhuri equation (4.43), we start with the definition (2.77) and write

$$
\begin{aligned}
\mathcal{J}^2 =: J_{bik}J_{alj}\eta^{ab}\delta^{ij}\delta^{lk} &= \left[g_{\mu\nu}\left(D_{i}E^{\mu}{}_{b}\right)N^{\nu}{}_{k}\right]\left[g_{\alpha\beta}\left(D_{l}E^{\alpha}{}_{a}\right)N^{\beta}{}_{j}\right]\eta^{ab}\delta^{ij}\delta^{lk}\\
&= g_{\mu\nu}g_{\alpha\beta}\left(\delta^{ij}N^{\rho}{}_{i}N^{\beta}{}_{j}\right)\left(\delta^{kl}N^{\gamma}{}_{l}N^{\nu}{}_{k}\right)\left[\eta^{ab}\left(D_{\gamma}E^{\alpha}{}_{a}\right)\left(D_{\rho}E^{\mu}{}_{b}\right)\right],
\end{aligned}
$$

then by eqs. (B.2) and (B.8),

$$
\begin{aligned}
J_{bik}J_{alj}\eta^{ab}\delta^{ij}\delta^{lk} &= -g_{\mu\nu}g_{\alpha\beta}\left(m^{\rho}\overline{m}^{\beta}+m^{\beta}\overline{m}^{\rho}\right)(m^{\gamma}\overline{m}^{\nu}+m^{\nu}\overline{m}^{\gamma})\\
&\times \left[\left(D_{\gamma}l^{\alpha}\right)\left(D_{\rho}n^{\mu}\right)+\left(D_{\gamma}n^{\alpha}\right)\left(D_{\rho}l^{\mu}\right)\right]\\
&= -[\langle D_{\mathbf{m}}\mathbf{n},\overline{\mathbf{m}}\rangle\langle D_{\mathbf{m}}\mathbf{l},\overline{\mathbf{m}}\rangle+\langle D_{\mathbf{m}}\mathbf{n},\overline{\mathbf{m}}\rangle\langle D_{\mathbf{m}}\mathbf{l},\overline{\mathbf{m}}\rangle]\\
&- [\langle D_{\mathbf{m}}\mathbf{n},\mathbf{m}\rangle\langle D_{\overline{\mathbf{m}}}\mathbf{l},\overline{\mathbf{m}}\rangle+\langle D_{\mathbf{m}}\mathbf{l},\mathbf{m}\rangle\langle D_{\overline{\mathbf{m}}}\mathbf{n},\overline{\mathbf{m}}\rangle]\\
&- [\langle D_{\overline{\mathbf{m}}}\mathbf{n},\overline{\mathbf{m}}\rangle\langle D_{\mathbf{m}}\mathbf{l},\mathbf{m}\rangle+\langle D_{\mathbf{m}}\mathbf{n},\mathbf{m}\rangle\langle D_{\overline{\mathbf{m}}}\mathbf{l},\overline{\mathbf{m}}\rangle]\\
&- [\langle D_{\overline{\mathbf{m}}}\mathbf{l},\mathbf{m}\rangle\langle D_{\overline{\mathbf{m}}}\mathbf{n},\mathbf{m}\rangle+\langle D_{\overline{\mathbf{m}}}\mathbf{l},\mathbf{m}\rangle\langle D_{\overline{\mathbf{m}}}\mathbf{n},\mathbf{m}\rangle].
\end{aligned}
$$

Finally, by using eqs. (A.11) and (A.14) we obtain

$$J_{bik}J_{alj}\eta^{ab}\delta^{ij}\delta^{lk} = 2\left(\mu\overline{\rho}+\overline{\mu}\rho+\sigma\lambda+\overline{\sigma}\overline{\lambda}\right). \tag{B.21}$$

B.5 Derivation of $\mathcal{K}^2$

The third term that appears on the right hand side of the Raychaudhuri equation (4.43), is obtained as the following once the definition (2.74) is considered.

$$\begin{aligned}\mathcal{K}^2 := K_{bci}K_{adj}\eta^{ab}\eta^{cd}\delta^{ij} &= \left[-g_{\mu\nu}\left(D_bE^{\mu}{}_c\right)N^{\nu}{}_i\right]\left[-g_{\alpha\beta}\left(D_aE^{\alpha}{}_d\right)N^{\beta}{}_j\right]\eta^{ab}\eta^{cd}\delta^{ij}\\ &= g_{\mu\nu}g_{\alpha\beta}\left(\delta^{ij}N^{\nu}{}_iN^{\beta}{}_j\right)\left(\eta^{ab}E^{\rho}{}_bE^{\gamma}{}_a\right)\left[\eta^{cd}\left(D_\rho E^{\mu}{}_c\right)\left(D_\gamma E^{\alpha}{}_d\right)\right].\end{aligned}$$

Also by making use of eqs. (B.1), (B.2) and (B.8) we write

$$\begin{aligned}K_{bci}K_{adj}\eta^{ab}\eta^{cd}\delta^{ij} &= \left(m^{\nu}\overline{m}^{\beta}+m^{\beta}\overline{m}^{\nu}\right)\left(l^{\rho}n^{\gamma}+l^{\gamma}n^{\rho}\right)\\ &\times \left[\left(D_\rho l^{\mu}\right)\left(D_\gamma n^{\alpha}\right)+\left(D_\rho n^{\mu}\right)\left(D_\gamma l^{\alpha}\right)\right]\\ &= \left[\langle D_{\mathbf{l}}\mathbf{l},\mathbf{m}\rangle\langle D_{\mathbf{n}}\mathbf{n},\overline{\mathbf{m}}\rangle+\langle D_{\mathbf{l}}\mathbf{n},\mathbf{m}\rangle\langle D_{\mathbf{n}}\mathbf{l},\overline{\mathbf{m}}\rangle\right]\\ &+ \left[\langle D_{\mathbf{n}}\mathbf{l},\mathbf{m}\rangle\langle D_{\mathbf{l}}\mathbf{n},\overline{\mathbf{m}}\rangle+\langle D_{\mathbf{n}}\mathbf{n},\mathbf{m}\rangle\langle D_{\mathbf{l}}\mathbf{l},\overline{\mathbf{m}}\rangle\right]\\ &+ \left[\langle D_{\mathbf{l}}\mathbf{l},\overline{\mathbf{m}}\rangle\langle D_{\mathbf{n}}\mathbf{n},\mathbf{m}\rangle+\langle D_{\mathbf{l}}\mathbf{n},\overline{\mathbf{m}}\rangle\langle D_{\mathbf{n}}\mathbf{l},\mathbf{m}\rangle\right]\\ &+ \left[\langle D_{\mathbf{n}}\mathbf{l},\overline{\mathbf{m}}\rangle\langle D_{\mathbf{l}}\mathbf{n},\mathbf{m}\rangle+\langle D_{\mathbf{n}}\mathbf{n},\overline{\mathbf{m}}\rangle\langle D_{\mathbf{l}}\mathbf{l},\mathbf{m}\rangle\right].\end{aligned}$$

Then by eqs. (A.9), (A.10), (A.12) and (A.13) we obtain the final form as

$$K_{bci}K_{adj}\eta^{ab}\eta^{cd}\delta^{ij} = -2\left(\kappa\nu+\overline{\kappa}\overline{\nu}+\pi\tau+\overline{\pi}\overline{\tau}\right). \tag{B.22}$$

B.6 Derivation of $\mathcal{R}_{\mathcal{W}}$

Now we derive the last term on the right hand side of the Raychaudhuri equation (4.43), in terms of the variables of the Newman-Penrose formalism, i.e.,

$$\begin{aligned}\mathcal{R}_{\mathcal{W}} = g(R(E_b,N_i)E_a,N_j)\eta^{ab}\delta^{ij} &= R_{\alpha\beta\mu\nu}E^{\mu}{}_bN^{\nu}{}_iE^{\beta}{}_aN^{\alpha}{}_j\eta^{ab}\delta^{ij}\\ &= R_{\alpha\beta\mu\nu}\left(\eta^{ab}E^{\mu}{}_bE^{\beta}{}_a\right)\left(\delta^{ij}N^{\nu}{}_iN^{\alpha}{}_j\right).\end{aligned}$$

Then by using eqs. (B.1) and (B.2) we obtain

$$\begin{aligned} g(R(E_b,N_i)E_a,N_j)\eta^{ab}\delta^{ij} &= -R_{\alpha\beta\mu\nu}\left(l^\mu n^\beta + l^\beta n^\mu\right)(m^\nu \overline{m}^\alpha + m^\alpha \overline{m}^\nu) \\ &= -\left[R_{\overline{\mathbf{m}}\mathbf{nlm}} + R_{\mathbf{mnl}\overline{\mathbf{m}}} + R_{\overline{\mathbf{m}}\mathbf{lnm}} + R_{\mathbf{mln}\overline{\mathbf{m}}}\right]. \end{aligned} \tag{B.23}$$

Since, the Riemann tensor is defined as,

$$R_{\mathbf{xyvw}} = -\left\langle D_{\mathbf{x}}D_{\mathbf{y}}\mathbf{v},\mathbf{w}\right\rangle + \left\langle D_{\mathbf{y}}D_{\mathbf{x}}\mathbf{v},\mathbf{w}\right\rangle + \left\langle D_{[\mathbf{x},\mathbf{y}]}\mathbf{v},\mathbf{w}\right\rangle,$$

we write

$$\begin{aligned} g(R(E_b,N_i)E_a,N_j)\eta^{ab}\delta^{ij} &= -\left[R_{\overline{\mathbf{m}}\mathbf{nlm}} + R_{\mathbf{mnl}\overline{\mathbf{m}}} + R_{\overline{\mathbf{m}}\mathbf{lnm}} + R_{\mathbf{mln}\overline{\mathbf{m}}}\right] \\ &= -\left[-\langle D_{\overline{\mathbf{m}}}D_{\mathbf{n}}\mathbf{l},\mathbf{m}\rangle + \langle D_{\mathbf{n}}D_{\overline{\mathbf{m}}}\mathbf{l},\mathbf{m}\rangle + \left\langle D_{[\overline{\mathbf{m}},\mathbf{n}]}\mathbf{l},\mathbf{m}\right\rangle\right] \\ &- \left[-\langle D_{\mathbf{m}}D_{\mathbf{n}}\mathbf{l},\overline{\mathbf{m}}\rangle + \langle D_{\mathbf{n}}D_{\mathbf{m}}\mathbf{l},\overline{\mathbf{m}}\rangle + \langle D_{[\mathbf{m},\mathbf{n}]}\mathbf{l},\overline{\mathbf{m}}\rangle\right] \\ &- \left[-\langle D_{\overline{\mathbf{m}}}D_{\mathbf{l}}\mathbf{n},\mathbf{m}\rangle + \langle D_{\mathbf{l}}D_{\overline{\mathbf{m}}}\mathbf{n},\mathbf{m}\rangle + \left\langle D_{[\overline{\mathbf{m}},\mathbf{l}]}\mathbf{n},\mathbf{m}\right\rangle\right] \\ &- \left[-\langle D_{\mathbf{m}}D_{\mathbf{l}}\mathbf{n},\overline{\mathbf{m}}\rangle + \langle D_{\mathbf{l}}D_{\mathbf{m}}\mathbf{n},\overline{\mathbf{m}}\rangle + \langle D_{[\mathbf{m},\mathbf{l}]}\mathbf{n},\overline{\mathbf{m}}\rangle\right]. \end{aligned} \tag{B.24}$$

Now we will make use of the commutation relations, (A.44) and (A.45), in order to write the inner products that involve the brackets in terms of the Newman-Penrose variables. In particular,

$$\begin{aligned} D_{[\overline{\mathbf{m}},\mathbf{n}]}\mathbf{l} &= -\nu D_{\mathbf{l}}\mathbf{l} - \left(\alpha + \overline{\beta} - \overline{\tau}\right)D_{\mathbf{n}}\mathbf{l} - (\overline{\gamma} - \gamma - \overline{\mu})D_{\overline{\mathbf{m}}}\mathbf{l} + \lambda D_{\mathbf{m}}\mathbf{l}, \\ D_{[\mathbf{m},\mathbf{n}]}\mathbf{l} &= -\overline{\nu} D_{\mathbf{l}}\mathbf{l} - (\overline{\alpha} + \beta - \tau)D_{\mathbf{n}}\mathbf{l} - (\gamma - \overline{\gamma} - \mu)D_{\mathbf{m}}\mathbf{l} + \overline{\lambda} D_{\overline{\mathbf{m}}}\mathbf{l}, \\ D_{[\overline{\mathbf{m}},\mathbf{l}]}\mathbf{n} &= -\left(\pi - \alpha - \overline{\beta}\right)D_{\mathbf{l}}\mathbf{n} + \overline{\kappa} D_{\mathbf{n}}\mathbf{n} - (\overline{\varepsilon} - \varepsilon + \rho)D_{\overline{\mathbf{m}}}\mathbf{n} - \overline{\sigma} D_{\mathbf{m}}\mathbf{n}, \\ D_{[\mathbf{m},\mathbf{l}]}\mathbf{n} &= -(\overline{\pi} - \overline{\alpha} - \beta)D_{\mathbf{l}}\mathbf{n} + \kappa D_{\mathbf{n}}\mathbf{n} - (\varepsilon - \overline{\varepsilon} + \overline{\rho})D_{\mathbf{m}}\mathbf{n} - \sigma D_{\overline{\mathbf{m}}}\mathbf{n}. \end{aligned} \tag{B.25}$$

At the next step of our derivation we make use of the propagation equations (A.9), (A.10), (A.11), (A.12), (A.13) and (A.14). Then we obtain

$$\begin{aligned} &\left\langle D_{[\overline{\mathbf{m}},\mathbf{n}]}\mathbf{l},\mathbf{m}\right\rangle + \langle D_{[\mathbf{m},\mathbf{n}]}\mathbf{l},\overline{\mathbf{m}}\rangle + \left\langle D_{[\overline{\mathbf{m}},\mathbf{l}]}\mathbf{n},\mathbf{m}\right\rangle + \langle D_{[\mathbf{m},\mathbf{l}]}\mathbf{n},\overline{\mathbf{m}}\rangle = \ldots \\ &\ldots = 2(\kappa\nu + \overline{\kappa\nu}) - 2(\tau\overline{\tau} + \pi\overline{\pi}) - 2\left(\rho\overline{\mu} + \overline{\rho}\mu + \lambda\sigma + \overline{\lambda\sigma}\right) \\ &+ \left[(\overline{\alpha} + \beta)(\pi + \overline{\tau}) + \left(\alpha + \overline{\beta}\right)(\overline{\pi} + \tau)\right] - \left[(\varepsilon - \overline{\varepsilon})(\mu - \overline{\mu}) + (\gamma - \overline{\gamma})(\rho - \overline{\rho})\right], \end{aligned}$$

so that

$$\begin{aligned}
g(R(E_b,N_i)E_a,N_j)\eta^{ab}\delta^{ij} &= [\langle D_{\overline{\mathbf{m}}}D_{\mathbf{n}}\mathbf{l},\mathbf{m}\rangle - \langle D_{\mathbf{n}}D_{\overline{\mathbf{m}}}\mathbf{l},\mathbf{m}\rangle] \\
&+ [\langle D_{\mathbf{m}}D_{\mathbf{n}}\mathbf{l},\overline{\mathbf{m}}\rangle - \langle D_{\mathbf{n}}D_{\mathbf{m}}\mathbf{l},\overline{\mathbf{m}}\rangle] \\
&+ [\langle D_{\overline{\mathbf{m}}}D_{\mathbf{l}}\mathbf{n},\mathbf{m}\rangle - \langle D_{\mathbf{l}}D_{\overline{\mathbf{m}}}\mathbf{n},\mathbf{m}\rangle] \\
&+ [\langle D_{\mathbf{m}}D_{\mathbf{l}}\mathbf{n},\overline{\mathbf{m}}\rangle - \langle D_{\mathbf{l}}D_{\mathbf{m}}\mathbf{n},\overline{\mathbf{m}}\rangle] \\
&- 2(\kappa\nu+\overline{\kappa\nu}) + 2(\tau\overline{\tau}+\pi\overline{\pi}) + 2\left(\rho\overline{\mu}+\overline{\rho}\mu+\lambda\sigma+\overline{\lambda}\overline{\sigma}\right) \\
&- \left[(\overline{\alpha}+\beta)(\pi+\overline{\tau})+\left(\alpha+\overline{\beta}\right)(\overline{\pi}+\tau)\right] \\
&+ [(\varepsilon-\overline{\varepsilon})(\mu-\overline{\mu})+(\gamma-\overline{\gamma})(\rho-\overline{\rho})].
\end{aligned}$$

Now we further use eqs. (A.10), (A.11), (A.12), (A.14), (A.15), (A.16) and (A.18) and write

$$\begin{aligned}
g(R(E_b,N_i)E_a,N_j)\eta^{ab}\delta^{ij} &= D_{\mathbf{m}}(\pi-\overline{\tau}) + D_{\overline{\mathbf{m}}}(\overline{\pi}-\tau) \\
&- \left[\left(\alpha-\overline{\beta}\right)(\overline{\pi}-\tau)+(\overline{\alpha}-\beta)(\pi-\overline{\tau})\right] \\
&- [(\varepsilon+\overline{\varepsilon})(\mu+\overline{\mu})+(\gamma+\overline{\gamma})(\rho+\overline{\rho})] \\
&+ [D_{\mathbf{n}}(\rho+\overline{\rho}) - D_{\mathbf{l}}(\mu+\overline{\mu})] \\
&+ \left[(\overline{\alpha}+\beta)(\pi+\overline{\tau})+\left(\alpha+\overline{\beta}\right)(\overline{\pi}+\tau)\right] \\
&- [(\varepsilon-\overline{\varepsilon})(\mu-\overline{\mu})+(\gamma-\overline{\gamma})(\rho-\overline{\rho})] \\
&- 2(\kappa\nu+\overline{\kappa\nu}) + 2(\tau\overline{\tau}+\pi\overline{\pi}) + 2\left(\rho\overline{\mu}+\overline{\rho}\mu+\lambda\sigma+\overline{\lambda}\overline{\sigma}\right) \\
&- \left[(\overline{\alpha}+\beta)(\pi+\overline{\tau})+\left(\alpha+\overline{\beta}\right)(\overline{\pi}+\tau)\right] \\
&+ [(\varepsilon-\overline{\varepsilon})(\mu-\overline{\mu})+(\gamma-\overline{\gamma})(\rho-\overline{\rho})].
\end{aligned}$$

Hence,

$$\begin{aligned}
g(R(E_b,N_i)E_a,N_j)\eta^{ab}\delta^{ij} &= D_{\mathbf{n}}(\rho+\overline{\rho}) - D_{\mathbf{l}}(\mu+\overline{\mu}) + D_{\mathbf{m}}(\pi-\overline{\tau}) + D_{\overline{\mathbf{m}}}(\overline{\pi}-\tau) \\
&- \left[\left(\alpha-\overline{\beta}\right)(\overline{\pi}-\tau)+(\overline{\alpha}-\beta)(\pi-\overline{\tau})\right] \\
&- [(\varepsilon+\overline{\varepsilon})(\mu+\overline{\mu})+(\gamma+\overline{\gamma})(\rho+\overline{\rho})] \\
&- 2(\kappa\nu+\overline{\kappa\nu}) + 2(\tau\overline{\tau}+\pi\overline{\pi}) + 2\left(\rho\overline{\mu}+\overline{\rho}\mu+\lambda\sigma+\overline{\lambda}\overline{\sigma}\right).
\end{aligned}$$

B.7 Alternative derivation of $\mathcal{R}_{\mathcal{W}}$

Here we will present a derivation of $\mathcal{R}_{\mathcal{W}}$ by using the decomposition of the Riemann tensor into its fully traceless, $C_{\mu\nu\alpha\beta}$, semi-traceless, $Y_{\mu\nu\alpha\beta}$, and the trace parts, $S_{\mu\nu\alpha\beta}$. For a 4-dimensional spacetime, the decomposition is as follows [12],

$$R_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta} + Y_{\mu\nu\alpha\beta} - S_{\mu\nu\alpha\beta}, \tag{B.26}$$

where $C_{\mu\nu\alpha\beta}$ is the Weyl tensor, R is the Ricci scalar of the spacetime and

$$\begin{aligned} Y_{\mu\nu\alpha\beta} &= \frac{1}{2}\left(g_{\mu\alpha}R_{\beta\nu} - g_{\mu\beta}R_{\alpha\nu} - g_{\nu\alpha}R_{\beta\mu} + g_{\nu\beta}R_{\alpha\mu}\right), && \text{(B.27)}\\ S_{\mu\nu\alpha\beta} &= \frac{R}{6}\left(g_{\mu\alpha}g_{\beta\nu} - g_{\mu\beta}g_{\alpha\nu}\right). && \text{(B.28)} \end{aligned}$$

The term we are after follows as

$$\begin{aligned} \mathcal{R}_{\mathcal{W}} = g(R(E_b,N_i)E_a,N_j)\eta^{ab}\delta^{ij} &= R_{\alpha\beta\mu\nu}E^{\mu}{}_{b}N^{\nu}{}_{i}E^{\beta}{}_{a}N^{\alpha}{}_{j}\eta^{ab}\delta^{ij} \\ &= R_{\alpha\beta\mu\nu}\left(\eta^{ab}E^{\mu}{}_{b}E^{\beta}{}_{a}\right)\left(\delta^{ij}N^{\nu}{}_{i}N^{\alpha}{}_{j}\right). \end{aligned}$$

Now by using eqs. (B.1) and (B.2) we obtain

$$\begin{aligned} g(R(E_b,N_i)E_a,N_j)\eta^{ab}\delta^{ij} &= -R_{\alpha\beta\mu\nu}\left(l^{\mu}n^{\beta} + l^{\beta}n^{\nu}\right)\left(m^{\nu}\overline{m}^{\alpha} + m^{\alpha}\overline{m}^{\nu}\right) \\ &= -\left[R_{\overline{\mathbf{m}}\mathbf{nlm}} + R_{\mathbf{mnl}\overline{\mathbf{m}}} + R_{\overline{\mathbf{m}}\mathbf{lnm}} + R_{\mathbf{mln}\overline{\mathbf{m}}}\right]. \end{aligned}$$

Symmetries of $R_{\mu\nu\alpha\beta}$ allows us to write

$$g(R(E_b,N_i)E_a,N_j)\eta^{ab}\delta^{ij} = -2\left(R_{\overline{\mathbf{m}}\mathbf{nlm}} + R_{\overline{\mathbf{m}}\mathbf{lnm}}\right),$$

and by using the decomposition (B.26),

$$\begin{aligned} g(R(E_b,N_i)E_a,N_j)\eta^{ab}\delta^{ij} &= -2\left(C_{\overline{\mathbf{m}}\mathbf{nlm}} + C_{\overline{\mathbf{m}}\mathbf{lnm}}\right) \\ &- 2\left(Y_{\overline{\mathbf{m}}\mathbf{nlm}} + Y_{\overline{\mathbf{m}}\mathbf{lnm}} - S_{\overline{\mathbf{m}}\mathbf{nlm}} - S_{\overline{\mathbf{m}}\mathbf{lnm}}\right). \end{aligned}$$

Here we make use of the symmetries of $C_{\mu\nu\alpha\beta}$ and the definition (A.22) to get

$$g(R(E_b,N_i)E_a,N_j)\eta^{ab}\delta^{ij} = -2\left(\Psi_2 + \overline{\Psi}_2\right) - 2\left(Y_{\overline{\mathbf{m}}\mathbf{nlm}} + Y_{\overline{\mathbf{m}}\mathbf{lnm}} - S_{\overline{\mathbf{m}}\mathbf{nlm}} - S_{\overline{\mathbf{m}}\mathbf{lnm}}\right). \tag{B.29}$$

By using the definitions of $Y_{\mu\nu\alpha\beta}$ and $S_{\mu\nu\alpha\beta}$ given in (B.27) and (B.28) we write

$$Y_{\overline{\mathbf{m}}\mathbf{n}\mathbf{l}\mathbf{m}} = \frac{1}{2}\left(\langle\overline{\mathbf{m}},\mathbf{l}\rangle R_{\mathbf{mn}} - \langle\overline{\mathbf{m}},\mathbf{m}\rangle R_{\mathbf{ln}} - \langle\mathbf{n},\mathbf{l}\rangle R_{\mathbf{m}\overline{\mathbf{m}}} + \langle\mathbf{n},\mathbf{m}\rangle R_{\mathbf{l}\overline{\mathbf{m}}}\right), \tag{B.30}$$

$$Y_{\overline{\mathbf{m}}\mathbf{l}\mathbf{n}\mathbf{m}} = \frac{1}{2}\left(\langle\overline{\mathbf{m}},\mathbf{n}\rangle R_{\mathbf{ml}} - \langle\overline{\mathbf{m}},\mathbf{m}\rangle R_{\mathbf{nl}} - \langle\mathbf{l},\mathbf{n}\rangle R_{\mathbf{m}\overline{\mathbf{m}}} + \langle\mathbf{l},\mathbf{m}\rangle R_{\mathbf{n}\overline{\mathbf{m}}}\right), \tag{B.31}$$

$$S_{\overline{\mathbf{m}}\mathbf{n}\mathbf{l}\mathbf{m}} = \frac{R}{6}\left(\langle\overline{\mathbf{m}},\mathbf{l}\rangle\langle\mathbf{m},\mathbf{n}\rangle - \langle\overline{\mathbf{m}},\mathbf{m}\rangle\langle\mathbf{l},\mathbf{n}\rangle\right), \tag{B.32}$$

$$S_{\overline{\mathbf{m}}\mathbf{l}\mathbf{n}\mathbf{m}} = \frac{R}{6}\left(\langle\overline{\mathbf{m}},\mathbf{n}\rangle\langle\mathbf{m},\mathbf{l}\rangle - \langle\overline{\mathbf{m}},\mathbf{m}\rangle\langle\mathbf{n},\mathbf{l}\rangle\right). \tag{B.33}$$

Also, since the Ricci scalar is $R = g^{\mu\nu}R_{\mu\nu} = 2\left(-R_{\mathbf{ln}} + R_{\mathbf{m}\overline{\mathbf{m}}}\right)$ and the Ricci tensor is a symmetric, we have

$$\mathcal{R}_{\mathcal{W}} = -2\left(\Psi_2 + \overline{\Psi_2}\right) - 2\left(\frac{R}{2} - \frac{R}{3}\right). \tag{B.34}$$

In the NP formalism one defines a variable $\Lambda = R/24$, thus we conclude that

$$\mathcal{R}_{\mathcal{W}} = -2\left(\Psi_2 + \overline{\Psi_2} + 4\Lambda\right). \tag{B.35}$$

C Other derivations

C.1 Gauss equation of $\mathbb{S}$

For a 2-dimensional spacelike surface embedded in a 4-dimensional spacetime, the Gauss equation reads as [136],

$$g(R(N_k, N_l)N_j, N_i) = \mathcal{R}_{ijkl} - J_{aik}J_{bjl}\eta^{ab} + J_{ajk}J_{bil}\eta^{ab}. \tag{C.1}$$

When we contract eq. (C.1) with $\delta^{ik}\delta^{jl}$ we get

$$g(R(N^k, N^l)N_k, N_l) = \mathcal{R}_{\mathbb{S}} - H + \mathcal{J}^2, \tag{C.2}$$

where $\mathcal{R}_{\mathbb{S}}$ is the intrinsic curvature scalar of $\mathbb{S}$, $H = J_{aik}J_{bjl}\eta^{ab}\delta^{ik}\delta^{jl}$ is the extrinsic curvature scalar of $\mathbb{S}$ and $\mathcal{J}^2 = J_{ajk}J_{bil}\eta^{ab}\delta^{ik}\delta^{jl}$ is one of the variables that appear in the contracted Raychaudhuri equation. Then derivation of $g(R(N^k, N^l)N_k, N_l)$ in terms of the NP variables proceeds as follows.

$$\begin{aligned} g(R(N_k, N_l)N_j, N_i)\delta^{ik}\delta^{jl} &= R_{\alpha\beta\mu\nu}N^\mu{}_k N^\nu{}_l N^\beta{}_j N^\alpha{}_i \delta^{ik}\delta^{jl} = R_{ijkl}\delta^{ik}\delta^{jl} \\ &= R_{\alpha\beta\mu\nu}\left(N^\mu{}_k N^\alpha{}_i \delta^{ik}\right)\left(N^\nu{}_l N^\beta{}_j \delta^{jl}\right). \end{aligned}$$

Now considering the relation (B.2) we write

$$\begin{aligned} R_{ijkl}\delta^{ik}\delta^{jl} &= R_{\alpha\beta\mu\nu}(m^\mu\overline{m}^\alpha + m^\alpha\overline{m}^\mu)\left(m^\nu\overline{m}^\beta + m^\beta\overline{m}^\nu\right) \\ &= R_{\overline{\mathbf{m}}\overline{\mathbf{m}}\mathbf{m}\mathbf{m}} + R_{\overline{\mathbf{m}}\mathbf{m}\mathbf{m}\overline{\mathbf{m}}} + R_{\mathbf{m}\overline{\mathbf{m}}\overline{\mathbf{m}}\mathbf{m}} + R_{\mathbf{m}\mathbf{m}\overline{\mathbf{m}}\overline{\mathbf{m}}}, \end{aligned}$$

and by considering the symmetries of $R_{\mu\nu\alpha\beta}$ we obtain

$$R_{ijkl}\delta^{ik}\delta^{jl} = -2R_{\overline{\mathbf{m}}\mathbf{m}\overline{\mathbf{m}}\mathbf{m}}.$$

Now let us use the decomposition (B.26) and write

$$R_{ijkl}\delta^{ik}\delta^{jl} \quad = \quad -2\left(C_{\overline{\mathbf{m}}\mathbf{m}\overline{\mathbf{m}}\mathbf{m}} + Y_{\overline{\mathbf{m}}\mathbf{m}\overline{\mathbf{m}}\mathbf{m}} - S_{\overline{\mathbf{m}}\mathbf{m}\overline{\mathbf{m}}\mathbf{m}}\right), \tag{C.3}$$

where

$$\begin{aligned}
C_{\overline{\mathbf{m}}\mathbf{m}\overline{\mathbf{m}}\mathbf{m}} &= \Psi_2 + \overline{\Psi_2}, &(C.4)\\
Y_{\overline{\mathbf{m}}\mathbf{m}\overline{\mathbf{m}}\mathbf{m}} &= \frac{1}{2}\left(\langle\overline{\mathbf{m}},\overline{\mathbf{m}}\rangle R_{\mathbf{mm}} - \langle\overline{\mathbf{m}},\mathbf{m}\rangle R_{\overline{\mathbf{m}}\mathbf{m}} - \langle\mathbf{m},\overline{\mathbf{m}}\rangle R_{\mathbf{m}\overline{\mathbf{m}}} + \langle\mathbf{m},\mathbf{m}\rangle R_{\overline{\mathbf{m}}\overline{\mathbf{m}}}\right)\\
&= -R_{\mathbf{m}\overline{\mathbf{m}}}, &(C.5)\\
S_{\overline{\mathbf{m}}\mathbf{m}\overline{\mathbf{m}}\mathbf{m}} &= \frac{R}{6}\left(\langle\overline{\mathbf{m}},\overline{\mathbf{m}}\rangle\langle\mathbf{m},\mathbf{m}\rangle - \langle\overline{\mathbf{m}},\mathbf{m}\rangle\langle\mathbf{m},\overline{\mathbf{m}}\rangle\right) = -\frac{R}{6}. &(C.6)
\end{aligned}$$

Equation (C.4) follows from the fact that Weyl tensor is traceless. To see this, consider the following. For any pair of vectors $\{\mathbf{v}, \mathbf{w}\}$ one can write

$$g^{xy}C_{x\mathbf{v}y\mathbf{w}} = 0 = -C_{\mathbf{lvnw}} - C_{\mathbf{nvlw}} + C_{\mathbf{mv}\overline{\mathbf{m}}\mathbf{w}} + C_{\overline{\mathbf{m}}\mathbf{vmw}}. \tag{C.7}$$

Now let us set $\mathbf{v} = \mathbf{m}$, $\mathbf{w} = \overline{\mathbf{m}}$, then we obtain

$$\begin{aligned}
0 &= -C_{\mathbf{lmn}\overline{\mathbf{m}}} - C_{\mathbf{nml}\overline{\mathbf{m}}} + C_{\mathbf{mm}\overline{\mathbf{m}}\overline{\mathbf{m}}} + C_{\overline{\mathbf{m}}\mathbf{mm}\overline{\mathbf{m}}}\\
&= C_{\mathbf{lm}\overline{\mathbf{m}}\mathbf{n}} + C_{\mathbf{l}\overline{\mathbf{m}}\mathbf{mn}} + 0 - C_{\overline{\mathbf{m}}\mathbf{m}\overline{\mathbf{m}}\mathbf{m}}. \qquad (C.8)
\end{aligned}$$

Then by using the definition given in (A.22) we find

$$C_{\overline{\mathbf{m}}\mathbf{m}\overline{\mathbf{m}}\mathbf{m}} \quad = \quad \Psi_2 + \overline{\Psi_2}. \tag{C.9}$$

In order to rewrite eq. (C.5) in terms of the curvature scalars consider

$$R = 2\left(-R_{\mathbf{ln}} + R_{\mathbf{m}\overline{\mathbf{m}}}\right) \qquad and \qquad \Phi_{11} = \frac{1}{4}\left(R_{\mathbf{ln}} + R_{\mathbf{m}\overline{\mathbf{m}}}\right). \tag{C.10}$$

Then we write

$$R_{\mathbf{m}\overline{\mathbf{m}}} = \frac{R + 8\Phi_{11}}{4}. \tag{C.11}$$

Therefore, substitution of equations (C.4), (C.5) and (C.6) into the decomposition (C.3) yields

$$\begin{aligned}
g(R(N_k, N_l)N^l, N^k) &= -2\left[\left(\Psi_2 + \overline{\Psi_2}\right) - \left(\frac{R + 8\Phi_{11}}{4}\right) + \frac{R}{6}\right]\\
&= -2\left(\Psi_2 + \overline{\Psi_2} - 2\Lambda - 2\Phi_{11}\right). \qquad (C.12)
\end{aligned}$$

C.2 Boost invariance of quasilocal charges:

C.2.1 Transformation of $\tilde{\nabla}_{\mathbb{T}}\mathcal{J}$ under Type-III Lorentz transformations

Under a Type-III Lorentz transformation, the null vectors **l** and **n** transform according to the relations (A.108) and (A.109) respectively. The transformed spin coefficients, γ', μ', ρ' and ε' can be obtained via the relations (A.114), (A.115), (A.119) and (A.122) so that the transformation of the term $\tilde{\nabla}_{\mathbb{T}}\mathcal{J}$ in eq. (4.50) follows as

$$\begin{aligned}
\tilde{\nabla}_{\mathbb{T}}\mathcal{J}' &= 2\left(D_{\mathbf{n}'}\rho' - D_{\mathbf{l}'}\mu'\right) - 2\left[\left(\varepsilon' + \overline{\varepsilon'}\right)\mu' + \left(\gamma' + \overline{\gamma}'\right)\rho'\right] \\
&= 2\left[\frac{1}{a^2}D_{\mathbf{n}}\left(a^2\rho\right) - a^2 D_{\mathbf{l}}\left(\frac{1}{a^2}\mu\right)\right] \\
&- 2\left\{a^2\left(\varepsilon + D_{\mathbf{l}}\left[\ln a + i\theta\right]\right) + a^2\left(\overline{\varepsilon} + D_{\mathbf{l}}\left[\ln a - i\theta\right]\right)\right\}\frac{1}{a^2}\mu \\
&- 2\left\{\frac{1}{a^2}\left(\gamma + D_{\mathbf{n}}\left[\ln a + i\theta\right]\right) + \frac{1}{a^2}\left(\overline{\gamma} + D_{\mathbf{n}}\left[\ln a - i\theta\right]\right)\right\}a^2\rho \\
&= 2\left(D_{\mathbf{n}}\rho - D_{\mathbf{l}}\mu\right) - 2\left[\left(\varepsilon + \overline{\varepsilon}\right)\mu + \left(\gamma + \overline{\gamma}\right)\rho\right]. \qquad \text{(C.13)}
\end{aligned}$$

Therefore $\tilde{\nabla}_{\mathbb{T}}\mathcal{J}$ is invariant under a Type-III Lorentz transformation.

C.2.2 Transformation of $\tilde{\nabla}_{\mathbb{S}}\mathcal{K}$ under Type-III Lorentz transformations

By using eq. (4.51), the transformed $\tilde{\nabla}_{\mathbb{S}}\mathcal{K}$ can be written as

$$\tilde{\nabla}_{\mathbb{S}}\mathcal{K}' = 2\left[D_{\mathbf{m}'}\pi' - D_{\overline{\mathbf{m}}'}\tau'\right] - 2\left[\left(\overline{\alpha}' - \beta'\right)\pi' + \left(\alpha' - \overline{\beta}'\right)\overline{\pi}'\right], \qquad \text{(C.14)}$$

in which the transformations of the complex null vectors **m** and $\overline{\mathbf{m}}$ are given in relations (A.110) and (A.111) respectively. Also, the transformed spin coefficients τ', β', α' and π', are obtained via the relations (A.113), (A.117), (A.120) and (A.123) so that we have

$$
\begin{aligned}
\tilde{\nabla}_{\mathbb{S}}\mathcal{K}' &= 2\left[e^{2i\theta}D_{\mathbf{m}}\left(e^{-2i\theta}\pi\right)-e^{-2i\theta}D_{\overline{\mathbf{m}}}\left(e^{2i\theta}\tau\right)\right] \\
&- 2\left\{e^{2i\theta}\left(\overline{\alpha}+D_{\mathbf{m}}\left[\ln a-i\theta\right]\right)-e^{2i\theta}\left(\beta+D_{\mathbf{m}}\left[\ln a+i\theta\right]\right)\right\}e^{-2i\theta}\pi \\
&- 2\left\{e^{-2i\theta}\left(\alpha+D_{\overline{\mathbf{m}}}\left[\ln a+i\theta\right]\right)-e^{-2i\theta}\left(\overline{\beta}+D_{\overline{\mathbf{m}}}\left[\ln a-i\theta\right]\right)\right\}e^{2i\theta}\overline{\pi}.
\end{aligned}
$$

Now by further imposing our null tetrad condition, $\tau+\overline{\pi}=0$ on the above equation we obtain

$$
\tilde{\nabla}_{\mathbb{S}}\mathcal{K}' = 2\left[D_{\mathbf{m}}\pi-D_{\overline{\mathbf{m}}}\tau\right]-2\left[\left(\overline{\alpha}-\beta\right)\pi+\left(\alpha-\overline{\beta}\right)\overline{\pi}\right]. \tag{C.15}
$$

Then, $\tilde{\nabla}_{\mathbb{S}}\mathcal{K}$ transforms invariantly under the spin-boost transformation of the null tetrad.

C.2.3 Transformation of $\mathcal{J}^2$ under Type-III Lorentz transformations

The transformation of $\mathcal{J}^2$ follows from the definition (4.52) plus the transformation relations (A.115), (A.116), (A.118) and (A.119) of the spin coefficients μ', σ', λ' and ρ'. Then we write

$$
\begin{aligned}
\mathcal{J}^{2\prime} &= 4\mu'\rho'+2\left(\sigma'\lambda'+\overline{\sigma}'\overline{\lambda}'\right) \\
&= 4\left(a^{-2}\mu\right)\left(a^{2}\rho\right)+2\left[\left(a^{2}e^{4i\theta}\sigma\right)\left(a^{-2}e^{-4i\theta}\lambda\right)+\left(a^{2}e^{-4i\theta}\overline{\sigma}\right)\left(a^{-2}e^{4i\theta}\overline{\lambda}\right)\right] \\
&= 4\mu\rho+2\left(\sigma\lambda+\overline{\sigma}\overline{\lambda}\right).
\end{aligned} \tag{C.16}
$$

Therefore $\mathcal{J}^2$ transforms invariantly under the spin-boost transformation of the null tetrad.

C.2.4 Transformation of $\mathcal{K}^2$ under Type-III Lorentz transformations

By using eq. (4.53) as for the definition of $\mathcal{K}^2$ and considering relations (A.112), (A.113), (A.121) and (A.123) for the transformations of spin coefficients ν', τ', κ' and

π' we write

$$\begin{aligned}
\mathcal{K}^{2\prime} &= -2\left(\kappa'\nu' + \overline{\kappa}'\overline{\nu}'\right) + 2\left(\pi'\overline{\pi}' + \tau'\overline{\tau}'\right) \\
&= -2\left[\left(a^4 e^{2i\theta}\kappa\right)\left(a^{-4} e^{-2i\theta}\nu\right) + \left(a^4 e^{-2i\theta}\overline{\kappa}\right)\left(a^{-4} e^{2i\theta}\overline{\nu}\right)\right] \\
&+ 2\left[\left(e^{-2i\theta}\pi\right)\left(e^{2i\theta}\overline{\pi}\right) + \left(e^{2i\theta}\tau\right)\left(e^{-2i\theta}\overline{\tau}\right)\right] \\
&= -2\left(\kappa\nu + \overline{\kappa\nu}\right) + 2\left(\pi\overline{\pi} + \tau\overline{\tau}\right).
\end{aligned} \tag{C.17}$$

Thus $\mathcal{K}^2$ is also invariant under spin-boost transformations.

C.2.5 Transformation of $\mathcal{R}_{\mathcal{W}}$ under Type-III Lorentz transformations

The Weyl scalar Ψ_2 transforms invariantly under spin-boost transformations according to the relation (A.135). Moreover, the parameter $\Lambda = R/24$ is invariant under such a transformation since the Ricci scalar is unchanged. Therefore, following eq. (4.49), it is easy to see that

$$\mathcal{R}_{\mathcal{W}}{}' = -2\left(\psi_2' + \overline{\psi}_2' + 4\Lambda'\right) = -2\left(\psi_2 + \overline{\psi}_2 + 4\Lambda\right), \tag{C.18}$$

and $\mathcal{R}_{\mathcal{W}}$ is invariant under spin-boost transformations.